J. Math. Anal. Appl. 446 (2017) 786–800

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Derivation of Fokker–Planck equations for stochastic systems under excitation of multiplicative non-Gaussian white noise

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A R T I C L E I N F O

Article history:
Received 4 April 2016
Available online 13 September 2016
Submitted by U. Stadtmueller

Keywords:
Fokker–Planck equations
Non-Gaussian white noise
Lévy processes
Marcus stochastic differential equations
Probability density function for solution

A B S T R A C T

Fokker–Planck equations describe time evolution of probability densities of stochastic dynamical systems and play an important role in quantifying propagation and evolution of uncertainty. Although Fokker–Planck equations can be written explicitly for systems excited by Gaussian white noise, they have remained unknown in general for systems excited by multiplicative non-Gaussian white noise. In this paper, we derive explicit forms of Fokker–Planck equations for one dimensional systems modeled by Marcus stochastic differential equations under multiplicative non-Gaussian white noise. As examples to illustrate the theoretical results, the derived formula is used to obtain Fokker–Planck equations for nonlinear dynamical systems under excitation of (i) α-stable white noise; (ii) combined Gaussian and Poisson white noise, respectively.

1. Introduction and statement of the problem

Stochastic differential equations (SDEs) are ubiquitous in a vast variety of fields, ranging from biology and physical sciences to finance and social sciences [15,10]. The probability density function associated with the stochastic process governed by an SDE is one of the qualities fully characterizing the statistical behavior of the solution to the SDE. Fokker–Planck equation provides the evolution of probability density functions and is an important tool to study how uncertainties propagate and evolve in physical and engineering dynamical systems [12,15,10,7].

Dynamical systems excited by Gaussian white noise are often modeled by SDEs driven by Brownian motions (or Wiener processes). For SDEs driven by Brownian motions, there are explicit formulas to obtain the associated Fokker–Planck equations, regardless the SDEs are in sense of Itô or Stratonovich [15,10]. However, it is *not* the case for SDEs with non-Gaussian noises.

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http://dx.doi.org/10.1016/j.jmaa.2016.09.010
0022-247X/

Dynamical systems excited by non-Gaussian white noise are usually modeled by SDEs driven by non-Gaussian Lévy processes. The connection between the Lévy driven SDE and the related partial integro-differential equations has been extensively studied since [21] and has found wide applications in many areas including finance [4]. This connection has been extended into nonlinear cases recently, see [8] among others. For SDEs driven by Lévy processes, there are two popular definitions, i.e., they are defined in sense of Itô or in sense of Marcus [13,14,11,1]. Marcus SDEs are often appropriate models in engineering and scientific practice, because they preserve certain physical quantities such as energy [13,14,11,19,20]. It is recently shown [19] that Marcus SDE is equivalent to the well known Di Paola–Falsone SDE [5,6] which is widely used in engineering and physics [9,22,20]. Comparison of Marcus integral and Stratonovich integral is recently discussed in [3] for systems with jump noise.

Solutions of both Itô and Marcus SDEs often are Markov processes [1]. It is well known that Fokker–Planck equations for Markov processes require the adjoint operators of the infinitesimal generators of the Markov processes [1]. Unlike the Gaussian cases, Fokker–Planck equations for SDEs driven by non-Gaussian Lévy processes remain *unknown* due to the difficulty in obtaining the explicit expressions for the adjoint of the infinitesimal generators associated with such SDEs [1]. While Fokker–Planck equations for Itô SDEs driven by some particular Lévy processes have been discussed by many authors ([18,17] among others), the research on Fokker–Planck equations for Marcus SDEs has received much less attention. A recent result about Fokker–Planck equations for Marcus SDEs is presented in [18], where an explicit form of Fokker–Planck equations is derived for Marcus SDEs under the condition that coefficients of the noise terms are strictly nonzero (i.e., the coefficient does not change sign). However, it is an *open* problem what the Fokker–Planck equations are like for Marcus stochastic differential equations under general conditions.

Lévy processes are stochastic processes with properties of independent and stationary increments, as well as stochastically continuous sample paths [1,16]. Examples of Lévy processes include Brownian motions, compound Poisson processes, α-stable processes and others. A one-dimensional Lévy process $L(t)$, taking values in $\mathbb{R}$, is characterized by a drift $b \in \mathbb{R}$, a positive real number A and a Borel measure ν defined on $\mathbb{R}$ and concentrated on $\mathbb{R}\backslash\{0\}$. In fact, this measure ν satisfies the following condition [1]

$$\int_{\mathbb{R}\backslash\{0\}} (y^2 \wedge 1)\nu(\mathrm{d}y) < \infty, \tag{1}$$

where $y^2 \wedge 1$ represents the minimum of y^2 and 1. This measure ν is called a Lévy jump measure for the Lévy process $L(t)$. A Lévy process with the generating triplet (b, A, ν) has the Lévy–Itô decomposition

$$\mathrm{d}L(t) = b\mathrm{d}t + \mathrm{d}B(t) + \int_{|y|<1} y\tilde{N}(\mathrm{d}t, \mathrm{d}y) + \int_{|y|\geq 1} yN(\mathrm{d}t, \mathrm{d}y), \tag{2}$$

where $N(\mathrm{d}t, \mathrm{d}x)$ is the Poisson random measure, $\tilde{N}(\mathrm{d}t, \mathrm{d}x) = N(\mathrm{d}t, \mathrm{d}x) - \nu(\mathrm{d}x)\mathrm{d}t$ is the compensated Poisson random measure, and $B(t)$ is the Brownian motion (i.e., Wiener process) with variance A.

Different kinds of Lévy processes can be obtained by taking different triplet (b, A, ν). Just as a Gaussian white noise can be regarded as the formal derivative of a Brownian motion process, a non-Gaussian white noise can be regarded as the formal derivative of some non-Gaussian Lévy process.

We shall consider stochastic dynamical systems described by the following SDE in sense of Marcus,

$$\mathrm{d}X(t) = f(X(t))\mathrm{d}t + \sigma(X(t)) \diamond \mathrm{d}L(t), \tag{3}$$

where $X(t)$ is a scalar process, and $L(t)$ is the one-dimensional Lévy process with the generating triplet (b, A, ν). Note that we only consider one-dimensional case in this paper. The solution of equation (3) is interpreted as