J. Math. Anal. Appl. 446 (2017) 801–822

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Damped wave equation with a critical nonlinearity in higher space dimensions

Nakao Hayashi [a,*], Pavel I. Naumkin [b]

[a] Department of Mathematics, Graduate School of Science, Osaka University, Osaka, Toyonaka, 560-0043, Japan
[b] Instituto de Matemáticas, UNAM Campus Morelia, AP 61-3 (Xangari), Morelia CP 58089, Michoacán, Mexico

ARTICLE INFO

Article history:
Received 25 July 2015
Available online 9 September 2016
Submitted by P. Yao

Keywords:
Damped wave equation
Critical nonlinearity
Large time asymptotics
Higher space dimension

ABSTRACT

We study the Cauchy problem for nonlinear damped wave equations with a critical defocusing power nonlinearity $|u|^{\frac{2}{n}} u$, where n denotes the space dimension. For $n = 1, 2, 3$, global in time existence of small solutions was shown in [4]. In this paper, we generalize the results to any spatial dimension via the method of decomposition of the equation into the high and low frequency components under the assumption that the initial data are small and decay rapidly at infinity. Furthermore we present a sharp time decay estimate of solutions with a logarithmic correction.

1. Introduction

We study the large time asymptotics of solutions to the Cauchy problem for the nonlinear damped wave equation

$$\begin{cases} \partial_t^2 u + 2\partial_t u - \Delta u + |u|^{\frac{2}{n}} u = 0, \ x \in \mathbf{R}^n, \ t > 0 \\ u(0,x) = u_0(x), \ \partial_t u(0,x) = u_1(x), \ x \in \mathbf{R}^n, \end{cases} \tag{1.1}$$

in higher space dimensions $n \geq 4$.

The power of the nonlinearity $p_F = 1+\frac{2}{n}$ is well-known Fujita critical exponent (see [1]). In the subcritical case $p < p_F$ the solution may blow up in a finite time even for small initial data (see [7,13,16,22–24]). In the supercritical case $p > p_F$ there exists global solution, which asymptotically behaves as a heat kernel (see [2,5,6,8–12,15,17–21]).

* Corresponding author.
E-mail addresses: nhayashi@math.sci.osaka-u.ac.jp (N. Hayashi), pavelni@matmor.unam.mx (P.I. Naumkin).

http://dx.doi.org/10.1016/j.jmaa.2016.09.005
0022-247X/

In [4] or the book [5], the Cauchy problem (1.1) was considered in the case of $n = 1, 2, 3$. We obtained the following result. If the initial data $u_0 \in \mathbf{H}^{\delta,0} \cap \mathbf{H}^{0,\delta}$, $u_1 \in \mathbf{H}^{\delta-1,0} \cap \mathbf{H}^{-1,\delta}$ with $\delta > \frac{n}{2}$ are small and such that

$$\theta = \int_{\mathbf{R}^n} 2u_0(x) + u_1(x)\,dx > 0, \int_{\mathbf{R}^n} u_0(x)\,dx > 0,$$

then the Cauchy problem (1.1) has a unique global solution $u \in \mathbf{C}\left([0,\infty);\mathbf{H}^{\delta,0}\right)$ satisfying the following asymptotic property

$$\left\|u(t) - \theta G(t) g^{-\frac{n}{2}}(t)\right\|_{\mathbf{L}^p} \leq C g^{-1-\frac{n}{2}}(t) \langle t\rangle^{-\frac{n}{2}\left(1-\frac{1}{p}\right)} \log g(t)$$

for all $t > 0$, where $1 \leq p \leq \infty$, $g(t) = 1 + \varkappa \log\langle t\rangle$, $\varkappa = \frac{\theta^{\frac{2}{n}}}{n\pi}\left(\frac{n}{n+2}\right)^{\frac{n}{2}}$, $G(t,x) = (2\pi t)^{-\frac{n}{2}} e^{-\frac{|x|^2}{2t}}$ is the heat kernel. Note that the nonlinearity $|u|^{\frac{2}{n}} u$ in the Cauchy problem (1.1) has not sufficient regularity, so we can not apply the methods of [4] or [5] to the higher space dimensions $n \geq 4$. In the present paper we apply a different approach based on the direct decomposition of equation (1.1) into the high and low frequency parts. It is known from the previous works, that in the high frequency part the solution has an exponential time decay, so that the solution is a remainder in this part. In the low frequency part we decompose the nonlinear damped wave equation into a system of two equations with the first order time derivative. One of these equations has exponential time decay. Another one is responsible for the large time asymptotics of solutions which is similar to that of the nonlinear heat equation. Our method in this paper works well for any dimension and it makes a proof much simpler than the previous one.

To state our result precisely we introduce some notations. The usual Lebesgue space is denoted by $\mathbf{L}^p$, $1 \leq p \leq \infty$. Define by

$$\mathbf{H}^{l,m} = \left\{\phi \in \mathbf{L}^2; \left\|\langle x\rangle^m \langle i\nabla\rangle^l \phi(x)\right\|_{\mathbf{L}^2} < \infty\right\}$$

the weighted Sobolev space, where $\langle x\rangle = \sqrt{1+|x|^2}$, $\langle i\nabla\rangle = \sqrt{1-\Delta}$. We also use the notation $\mathbf{H}^l = \mathbf{H}^{l,0}$. We denote by $\mathcal{F}$ the Fourier transformation

$$\widehat{u}(\xi) \equiv \mathcal{F}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{-i\xi\cdot x} u(x)\,dx$$

and $\mathcal{F}^{-1}$ is the inverse Fourier transformation

$$\mathcal{F}^{-1}u = (2\pi)^{-\frac{n}{2}} \int_{\mathbf{R}^n} e^{i\xi\cdot x} u(\xi)\,d\xi.$$

By $\mathbf{C}(\mathbf{I};\mathbf{B})$ we denote the space of continuous functions from a time interval $\mathbf{I}$ to the Banach space $\mathbf{B}$. In what follows we denote by C different positive constants.

Our main result is the following.

Theorem 1.1. *Let the initial data* $u_0 \in \mathbf{H}^2 \cap \mathbf{H}^{1,\delta}$, $u_1 \in \mathbf{H}^1 \cap \mathbf{H}^{0,\delta}$ *with* $\delta > \frac{n}{2}$, $n \geq 4$ *and*

$$\theta = \int_{\mathbf{R}^n} 2u_0(x) + u_1(x)\,dx > 0.$$