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Hypercyclicity of shifts on weighted L^p spaces of directed trees

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In memory of Jaime Cruz Sampedro, mathematician, teacher, colleague, and friend

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ABSTRACT

In this paper, we study the hypercyclicity of forward and backward shifts on weighted L^p spaces of a directed tree. In the forward case, only the trivial trees may support hypercyclic shifts, in which case the classical results of Salas [21] apply. For the backward case, nontrivial trees may support hypercyclic shifts. We obtain necessary conditions and sufficient conditions for hypercyclicity of the backward shift and, in the case of a rooted tree on an unweighted space, we show that these conditions coincide.

1. Introduction

A bounded operator on a Banach space is called hypercyclic if there exists a vector such that its orbit under the operator is dense in the space. The study of hypercyclicity (in other types of spaces) can be traced back to the first half of the 20th century, to the papers of Birkhoff [6] and MacLane [17]. The first example of a hypercyclic operator on a Banach space was given by Rolewicz [20] in 1969, but it was not until the last two decades of the 20th century that the study of hypercyclicity really took off. Instead of giving here the detailed history of the advances in hypercyclicity in the past 35 years, we refer the reader to the excellent books by Grosse-Erdmann and Peris [15] and Bayart and Matheron [5], where the reader can find more information about this concept and its importance.

One large source of examples and counterexamples in the study of bounded operators is the class of weighted shifts. The study of weighted shifts was initiated in the now classical paper of Shields [23] and continued by many authors. The characterization of the hypercyclicity of weighted shifts is due to Salas [21]

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(see the books [5] and [15] for an alternative statement of this characterization). Many other classes of operators on Banach spaces have been shown to be hypercyclic, under certain conditions. Two other famous families of operators on Hilbert spaces that contain hypercyclic operators, are adjoints of multiplication operators (see [13]) and composition operators on spaces of holomorphic functions (see, e.g., [22]).

The interest of the study of operators on infinite trees is motivated mainly by the research in harmonic analysis dealing with the Laplace operator on discrete structures, perhaps initiated in the papers [7,8]. In particular, infinite trees can be seen as the natural discretizations of the hyperbolic disk. Much more information about these topics can be found in the papers [1–4,9–11,19]. We should mention that the paper [19] studies hypercyclicity for composition operators defined on the boundary of nondirected trees.

In [16], Jabłoński, Jung and Stochel initiated the study of weighted shifts on Hilbert spaces of functions defined on infinite directed trees. In their paper, they study many operator-theoretic properties of these operators, such as boundedness, hyponormality, subnormality and spectral properties.

Motivated by the work in [16], in this paper we study the hypercyclicity of shifts on directed trees on the weighted $\mathbf{L}^p$ space of a directed tree. In concrete, we show in Section 3 that "forward" shifts are never hypercyclic unless they reduce to the classical cases, in which the characterization by Salas mentioned above can be applied. In Section 4, we find a concrete form for the adjoint of the shift and define what we mean by a "backward shift". More interestingly, we show in Sections 5 and 6 that this backward shift on weighted directed trees, may be hypercyclic if some conditions are satisfied. In concrete, the main results of this paper provide necessary conditions and sufficient conditions for hypercyclicity of the backward shift in the case where the tree has a root. These two conditions coincide when the space is unweighted, in which case the hypercyclicity of the operator depends on a simple property of the tree, that of having no "free ends". In the case of the unrooted tree, we only give necessary conditions and show an example when these conditions are satisfied. When applied to the classical backward shifts, all of these conditions reduce to the ones obtained by Salas.

Before we begin, we should mention that in [16], the authors study the weighted shift on an unweighted $\mathbf{L}^2$ space of the tree. In this paper we prefer to concentrate on the unweighted shift on the weighted spaces $\mathbf{L}^p$ of the tree. The reason for this is that the results are cleaner in the case of the weighted space, as is also the case of the classical shifts. Results for the hypercyclicity of the weighted shift can be obtained by similarity with the shift of the weighted space, as it is done in, for example, [5,15]. We leave these results as an exercise for the interested reader.

2. Definitions and notation

In this section, we set the basic definitions and notation needed for the rest of the paper. We denote by $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{R}$, $\mathbb{R}_+$ and $\mathbb{C}$ the sets of natural numbers, the nonnegative integers, the integers, the real numbers, the positive real numbers, and the complex numbers, respectively.

Hypercyclicity. We first state the main definition in this paper and a few comments about it. After that, we present the main tool used to prove that operators are hypercyclic.

Definition 2.1. Let $\mathcal{B}$ be a Banach space and $S : \mathcal{B} \to \mathcal{B}$ a bounded operator. We say that S is hypercyclic if there exists a vector $x \in \mathcal{B}$ such that

$$\{S^n x \,:\, n \in \mathbb{N}_0\}$$

is dense in $\mathcal{B}$. The (necessarily) nonzero vector x is called a hypercyclic vector.

Observe that if the operator $S : \mathcal{B} \to \mathcal{B}$ is hypercyclic, the Banach space $\mathcal{B}$ is necessarily separable. Observe also that, if x is a hypercyclic vector for S, then, for each $n \in \mathbb{N}$, the vector $S^n x$ is also a