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Haar meager sets, their hulls, and relationship to compact sets

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ABSTRACT

Let G be an abelian Polish group. We show that there is a strongly Haar meager set in G without any F_σ Haar meager hull (and that this still remains true if we replace F_σ by any other class of the Borel hierarchy). We also prove that there is a coanalytic naively strongly Haar meager set without any Haar meager hull. Further, we investigate the relationship of the collection of all compact sets to the collection of all Haar meager sets in non-locally compact Polish groups.

1. Introduction

The notion of Haar meager sets (in abelian Polish groups) was introduced by Darji [4] (and straightforwardly generalized to the case of arbitrary Polish groups in [5]). Haar meager sets form a topological counterpart to the so called Haar null sets defined by Christensen in [3]. Let us recall the definitions (and some of its variants).

Definition 1. Let G be a Polish group. A set $A \subseteq G$ is said to be

(i) *Haar null* if there are a Borel set $B \subseteq G$ such that $A \subseteq B$, and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$;
(ii) *generalized Haar null* if there are a universally measurable set $B \subseteq G$ such that $A \subseteq B$, and a Borel probability measure μ on G such that $\mu(gBh) = 0$ for every $g, h \in G$.

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Definition 2. Let G be a Polish group. A set $A \subseteq G$ is said to be

(i) *Haar meager* if there are a Borel set $B \subseteq G$ such that $A \subseteq B$, a compact metric space K and a continuous function $f\colon K \to G$ such that $f^{-1}(gBh)$ is meager in K for every $g, h \in G$;
(ii) *strongly Haar meager* if there are a Borel set $B \subseteq G$ such that $A \subseteq B$, and a compact set $K \subseteq G$ such that $gBh \cap K$ is meager in K for every $g, h \in G$;
(iii) *naively Haar meager* if there is a compact metric space K and a continuous function $f\colon K \to G$ such that $f^{-1}(gAh)$ is meager in K for every $g, h \in G$;
(iv) *naively strongly Haar meager* if there is a compact set $K \subseteq G$ such that $gAh \cap K$ is meager in K for every $g, h \in G$.

Darji proved that in any abelian Polish group G, Haar meager sets form a σ-ideal contained in the σ-ideal of all meager sets, and that these two σ-ideals coincide if and only if G is locally compact. These results clearly correlate to the analogous well known results concerning Haar null sets proved by Christensen in [3]. Other similarities of the σ-ideals of Haar meager sets and of Haar null sets were investigated in [7] and [5]. We should note that it is not known whether every (naively) Haar meager set is (naively) strongly Haar meager.

In this paper we study two different topics. First, we investigate whether it is possible to replace the Borel hull B from (i) in Definition 2 by a hull from some other class of sets, e.g. by an F_σ hull. Next, we investigate the relationship of the collection of all compact sets to the collection of all Haar meager sets in non-locally compact Polish groups.

Let us look at the content of this paper a little closer. In Chapter 2 we introduce the notation and recall some facts needed later. Chapter 3 is devoted to the descriptive complexity of hulls of Haar meager sets. This chapter is very closely inspired by a paper of Elekes and Vidnyánszky [6] (both by the results and by the proof methods). Elekes and Vidnyánszky answered a question posed by Mycielski [10, Problem (P1)] by proving that in any non-locally compact abelian Polish group, there is a Haar null set which cannot be cover by a G_δ Haar null set [6, Theorem 1.3]. They also offered a more general statement where G_δ may be replaced by any other class of the Borel hierarchy [6, Theorem 1.4]. Finally, they answered a question asked by Fremlin at his webpage by showing that there are generalized Haar null sets which are not Haar null [6, Theorem 1.8]. Comparing Definitions 1 and 2, it is reasonable to believe that analogous results could be proved also for Haar meager sets where 'Haar null' corresponds to 'Haar meager' and the multiplicative class G_δ corresponds to the additive class F_σ. The purpose of Chapter 3 is showing that this is indeed possible by making only subtle changes in the proofs from [6]. The main result of Chapter 3 is Theorem 10 whose special cases are Theorems 12 and 13. Note that instead of the family of all universally measurable sets, the topological counterpart to the generalized Haar null sets should consider the family of all sets $B \subseteq G$ such that for every compact metric space K and every continuous function $f\colon K \to G$, the preimage $f^{-1}(B) \subseteq K$ has the Baire property. However, we do not need to formulate this definition (and its strong variant) since Theorem 13 uses a coanalytic naively strongly Haar meager set which is a stronger notion.

The results from Chapter 4 are inspired by the question posed in [5, Question 6] which asks whether compact subsets of a non-locally compact Polish group are Haar meager. We provide a sufficient condition for F_σ subsets of a Polish group to be strongly Haar meager. Then we show that if G is either the symmetric group S_∞ or any non-locally compact Polish group with a translation invariant metric then all compact subsets of G satisfy this sufficient condition, and thus they are strongly Haar meager. This improves the result by Jabłońska who proved that every compact subset of a non-locally compact abelian Polish group is Haar meager [7].