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A solution to the problem of Raşa connected with Bernstein polynomials

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ABSTRACT

During the *Conference on Ulam's Type Stability* (Rytro, Poland, 2014), Ioan Raşa recalled his 25-years-old problem concerning some inequality involving the Bernstein polynomials. We offer the complete solution (in positive). As a tool we use stochastic orderings (which we prove for binomial distributions) as well as so-called concentration inequality. Our methods allow us to pose (and solve) the extended version of the problem in question.

1. Introduction

The Bernstein fundamental polynomials of degree $n \in \mathbb{N}$ are given by the formula

$$b_{n,i}(x) = \binom{n}{i} x^i (1-x)^{n-i}\,, \quad i = 0, 1, \ldots, n\,.$$

In 2014, during the *Conference on Ulam's Type Stability* held in Rytro (Poland), Ioan Raşa recalled his 25-years-old problem [1, Problem 2, p. 164] related to the preservation of convexity by the Bernstein–Schnabl operators.

Problem. Prove or disprove that

$$\sum_{i,j=0}^{n} \big(b_{n,i}(x)b_{n,j}(x) + b_{n,i}(y)b_{n,j}(y) - 2b_{n,i}(x)b_{n,j}(y)\big) f\left(\frac{i+j}{2n}\right) \geqslant 0 \tag{1.1}$$

for each convex function $f \in \mathcal{C}\big([0,1]\big)$ and for all $x, y \in [0,1]$.

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The aim of this paper is to answer the above-stated problem affirmatively (i.e., to prove (1.1)).

Let us invoke some basic notations and results (see e.g. [3]). Let $(\Omega, \mathcal{F}, P)$ be a probability space. As usual, $F_X(x) = P(X < x)$ $(x \in \mathbb{R})$ stands for the probability distribution function of a random variable $X : \Omega \to \mathbb{R}$, while μ_X is the distribution corresponding to X. For real-valued random variables X, Y with finite expectations we say that X *is dominated by* Y *in the stochastic convex ordering sense*, if

$$\mathbb{E}\, f(X) \leqslant \mathbb{E}\, f(Y) \tag{1.2}$$

for all convex functions $f : \mathbb{R} \to \mathbb{R}$ (for which the expectations above exist). In that case we write $X \leqslant_{\mathrm{cx}} Y$ or $F_X \leqslant_{\mathrm{cx}} F_Y$.

The main idea of our solution is to study the convex stochastic ordering within the class of binomial distributions. To this end we make use of Ohlin's Lemma [8, Lemma 2, p. 256], which gives a sufficient condition for two random variables to be in the stochastic convex ordering relation.

Ohlin's Lemma. *Let X, Y be two random variables and suppose that $\mathbb{E}\, X = \mathbb{E}\, Y$. If the probability distribution functions F_X, F_Y cross exactly once, i.e.,*

$$F_X(x) \leqslant F_Y(x) \text{ if } x < x_0 \quad \text{and} \quad F_X(x) \geqslant F_Y(x) \text{ if } x > x_0$$

for some $x_0 \in \mathbb{R}$, then $X \leqslant_{\mathrm{cx}} Y$.

Originally this lemma was applied to certain insurance problems and it was lesser-known to mathematicians for a long time. It was re-discovered by the second-named author, who found a number of applications in the theory of convex functions (cf. [10,11]).

Remark 1. Szostok noticed in [12] that if the measures μ_X, μ_Y corresponding to X, Y, respectively, are concentrated on the interval $[a, b]$, then, in fact, the relation $X \leqslant_{\mathrm{cx}} Y$ holds if and only if the inequality (1.2) is satisfied for all continuous convex functions $f : [a, b] \to \mathbb{R}$.

Recall that $X \sim B(p)$ means that the random variable X has the Bernoulli distribution with the parameter $p \in (0, 1)$. If X has the binomial distribution with the parameters $n \in \mathbb{N}$ and $p \in (0, 1)$ (which we denote by $X \sim B(n, p)$ for short), then, of course,

$$P(X = k) = \binom{n}{k} p^k (1-p)^{n-k}, \quad k = 0, 1, \ldots, n \quad \text{and} \quad \mathbb{E}\, X = np. \tag{1.3}$$

Below we recall the binomial convex concentration inequality, which plays an important rôle in our considerations. It is, in fact, due to Hoeffding [4]. Nevertheless, Hoeffding did not state it in the form required for our purposes. The desired form can be found, *e.g.*, in [5, Proposition 1, p. 67].

Theorem 2. *Let $b_i \sim B(p_i)$ $(i = 1, \ldots, n)$ be independent random variables. Set $S_n = b_1 + \cdots + b_n$. Let $\overline{p} = \dfrac{p_1 + \cdots + p_n}{n}$ and suppose that $S_n^* \sim B(n, \overline{p})$. Then*

$$\mathbb{E}\, \Phi(S_n) \leqslant \mathbb{E}\, \Phi(S_n^*)$$

for any convex function $\Phi : \mathbb{R} \to \mathbb{R}$ (which means that $S_n \leqslant_{\mathrm{cx}} S_n^*$).

A crucial result required to solve Raşa's problem reads as follows.