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Sensitivity of dendrite maps

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ABSTRACT

Suppose that X is a dendrite and $f : X \to X$ is a sensitive continuous map. We show that (a) (X, f) contains a bilaterally transitive subsystem with nonempty interior; (b) the system (X, f) satisfies only one of the following two conditions: (b1) (X, f) contains a topologically transitive subsystem with nonempty interior; (b2) there exists an f-invariant nowhere dense closed subset A of X such that the attraction basin $\mathrm{Basin}(A, f)$ contains a residual subset B of an open set and the strong attraction basin $\mathrm{Sbasin}(A, f)$ is dense in B; (c) if X is completely regular, then (X, f) contains a relatively strongly mixing subsystem with nonempty interior, dense periodic points and positive topological entropy. Unlike for interval maps, we construct a sensitive dendrite map with zero topological entropy.

1. Introduction

A *topological dynamical system* is a pair (X, f) where X is a compact metric space and $f : X \to X$ is a continuous map. A system (X, f) is called *sensitive*, or f is called *sensitive* for simplicity, if there exists a constant $c > 0$, called a *sensitivity constant* of the system (X, f), such that for any nonempty open set $U \subset X$, there is $n \in \mathbb{N}$ such that $\mathrm{diam}(f^n(U)) > c$. Sensitivity is usually regarded as an important feature of chaotic systems, though nowadays there is no universal agreement on the definition of chaos. For example, sensitivity is a key ingredient in the definitions of Devaney chaos and Auslander–Yorke chaos (also called Ruelle–Takens chaos) (see [12,4]).

The relationships between sensitivity, topological transitivity, and topological entropy have been extensively studied. In [8] it is shown that a transitive system (X, f) with dense periodic points must be sensitive except that X is a finite set. This result was extended to transitive non-minimal systems with dense minimal points by Glasner and Weiss in [17]. Also, a transitive system with positive topological entropy must be

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sensitive (see [17]). Some simple examples can show that the converses of the above results are far from being true for general systems. However, the situation is completely different when we consider one-dimensional systems. It is known that sensitivity implies the existence of a transitive cycle of intervals for any interval map f (see [9]). This implies that f has positive topological entropy (see, e.g., [10]). Some stronger forms of sensitivity such as Li–Yorke sensitivity, strong sensitivity, syndetic sensitivity and cofinite sensitivity have been discussed in [3,32]. Sensitivity also played a key role in proving that Devaney's chaos implies Li–Yorke's chaos (see [19,24]).

The aim of this paper is to study sensitivity of dendrite maps. We mainly consider the relations between sensitivity, transitivity and topological entropy for dendrite maps. Recall that a *continuum* is a compact connected metric space, and a *dendrite* is a locally connected continuum containing no simple closed curves. By a *tree*, we mean a connected compact one-dimensional polyhedron which contains no simple closed curves. Clearly trees are dendrites by definition. Dynamical systems on dendrites appeared naturally in the study of complex dynamical systems and hyperbolic geometry. In recent years, many people started to study the dynamics of dendrite maps. Although dendrites possess many properties of trees, dynamical properties on dendrites are much more varied than that on trees. For example, it is well known that the $\overline{P} = \overline{R}$ property holds for trees (see [34]), but a counterexample for the Gehman dendrite was constructed by Kato in [21]. Further, Illanes proved that a dendrite X contains a Gehman dendrite if and only if X does not have the $\overline{P} = \overline{R}$ property in [20] (see also [6,25]). Recently, Hoehn and Mouron gave a map of the Wazewski's universal dendrite that is weakly mixing but not mixing (see [18]) and has a unique periodic point (see [1]), which also showed the sharp difference between the dynamics of tree maps and dendrite maps. One may refer to [5,15,22,29–31] for more results about the dynamics of dendrite maps.

Before the statement of the theorem, let us recall some definitions and notation. We denote by $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ the sets of real numbers, integers and positive integers respectively. Let (X, f) be a topological dynamical system. For $x \in X$, the set $O^+(x, f) = \{f^n(x) : n \in \mathbb{N} \cup \{0\}\}$ is called the *forward orbit* (or usually, the *orbit*) of x under f, and $O^-(x, f) = \cup\{f^{-n}(x) : n \in \mathbb{N} \cup \{0\}\}$ is called the *backward orbit* of x under f. The set $O(x, f) = O^+(x, f) \cup O^-(x, f)$ is called the *bilateral orbit* of x under f. We should note that $f^{-n}(x)$ may be a set with more than one point if f is not injective, so the symbol $O(x, f)$ has different meanings from its usual ones.

Recall that the *ω-limit set* $\omega(x, f)$ of a point $x \in X$ is the set of all limit points of $O^+(x, f)$, i.e., $\omega(x, f) = \{y \in X :$ there is a sequence of positive integers $n_i \to +\infty$ s.t. $f^{n_i}(x) \to y\}$. A point $x \in X$ is a *nonwandering point of* f if for every neighborhood U of x there is $n \in \mathbb{N}$ such that $f^n(U) \cap U \neq \emptyset$. The set of all nonwandering points of f is denoted by $\Omega(X, f)$. A subset A of X is called *f-invariant* provided that $f(A) \subset A$. If $A \subset X$ is closed and f-invariant, then $(A, f|_A)$ is also a topological dynamical system, which is called a *subsystem* of (X, f). The *attraction basin* of an f-invariant closed set A is defined to be the set $\mathrm{Basin}(A, f) = \{x \in X : \omega(x, f) \cap A \neq \emptyset\}$ and the *strong attraction basin* of A is defined to be the set $\mathrm{Sbasin}(A, f) = \{x \in \mathrm{Basin}(A, f) : f^n(x) \in A$ for some $n \in \mathbb{N}\}$. The set A is called the *attracting set* of its attraction basin. To compare the notion of attracting set in the present paper with various definitions of attractors, one may consult [26,27].

In this paper, we will refer to the following kinds of transitivity. Let (X, f) be a topological dynamical system, then

(1) (X, f) is said to be *topologically transitive*, or *transitive* in short, if for every pair of nonempty open subsets U and V of X, there is $n \in \mathbb{N}$, such that $f^n(U) \cap V \neq \emptyset$;
(2) (X, f) is said to be *point transitive*, if there exists a point $x \in X$, such that the closure $\overline{O^+(x, f)} = X$, and the point x is said to be a *transitive point*;
(3) (X, f) is said to be *bilaterally transitive*, if there exists a point $x \in X$, such that $\overline{O(x, f)} = X$;
(4) (X, f) is said to be *strongly mixing* if for any pair of nonempty open subsets U and V of X, there is some $N \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$ for all $n > N$.