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Regularity of partial differential operators in ultradifferentiable spaces and Wigner type transforms

Chiara Boiti [a,*], David Jornet [b], Alessandro Oliaro [c]

[a] *Dipartimento di Matematica e Informatica, Università di Ferrara, Via Machiavelli n. 30, I-44121 Ferrara, Italy*
[b] *Instituto Universitario de Matemática Pura y Aplicada IUMPA, Universitat Politècnica de València, Camino de Vera, s/n, E-46071 Valencia, Spain*
[c] *Dipartimento di Matematica, Università di Torino, Via Carlo Alberto n. 10, I-10123 Torino, Italy*

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ABSTRACT

We study the behaviour of linear partial differential operators with polynomial coefficients via a Wigner type transform. In particular, we obtain some results of regularity in the Schwartz space $\mathcal{S}$ and in the space $\mathcal{S}_\omega$ as introduced by Björck for weight functions ω. Several examples are discussed in this new setting.

1. Introduction

In this paper we are concerned with the regularity of linear partial differential operators with polynomial coefficients. This problem was introduced by Shubin [22], who says that a linear operator $A : \mathcal{S}' \to \mathcal{S}'$ is regular if the conditions $u \in \mathcal{S}'$, $Au \in \mathcal{S}$ imply that $u \in \mathcal{S}$. In [22, Chapter IV] global pseudodifferential operators on $\mathbb{R}^n$ are studied, giving a notion of (global) hypoellipticity (see formula (5.1)), that implies the above mentioned regularity in Schwartz spaces. Such global pseudodifferential operators are defined by treating in the same way variables and covariables, and have as basic examples linear partial differential operators with polynomial coefficients. The global hypoellipticity, on the other hand, is far from being a necessary condition for the regularity of an operator; some results have been obtained in this direction, we refer in particular to [23] who proved the regularity of the Twisted Laplacian (a non-hypoelliptic operator in two variables), and to [20], who gave a characterization of the regularity of ordinary differential operators in the case when the roots of the corresponding Weyl symbol are suitably separated at infinity. Moreover,

* Corresponding author.
E-mail addresses: chiara.boiti@unife.it (C. Boiti), djornet@mat.upv.es (D. Jornet), alessandro.oliaro@unito.it (A. Oliaro).

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in [9] a class of non-hypoelliptic regular partial differential operators with polynomial coefficients have been found, by using a technique related to transformations of Wigner type; such class includes as a particular case the Twisted Laplacian. The idea to use quadratic transformations for the study of general properties of partial differential equations (that underlies [9], as well as the present paper) goes back to some works related to engineering applications, cf. [13,15], where the main aim is to understand the Wigner transform of the solution of a partial differential equation without finding the solution itself; the ideas of [13,15] are developed and organically presented in [12]. In the present paper we study the regularity of linear partial differential operators, in the spirit of [9], developing the research in two directions; first, we consider a general representation in the Cohen class, defined as

$$Q[w] := \sigma * \mathrm{Wig}[w]$$

for a kernel $\sigma \in \mathcal{S}'$, where $\mathrm{Wig}[w]$ is the Wigner transform, defined as

$$\mathrm{Wig}[w](x, y) := \int e^{-ity} w\left(x + \frac{1}{2}t, x - \frac{1}{2}t\right) dt.$$

The idea is that a linear partial differential operator B with polynomial coefficients is transformed into another one by a formula of the kind

$$Q[Bw] = \tilde{B}Q[w];$$

moreover, under suitable hypotheses on the kernel σ, the regularity is preserved by such transformation, and if we start from a global hypoelliptic operator B we find in general a non-global hypoelliptic operator $\tilde{B}$. Then, we can construct a large class of partial differential operators that are regular but not globally hypoelliptic. We also study regularity and the results just mentioned for the class $\mathcal{S}_\omega$ for a weight function ω, as introduced by Björck [2] (see also [14] for non-subadditive weight functions), which gives a large scale of examples, working in particular for Gevrey weight functions. This requires a preliminary study of the Schwartz ultradifferentiable space $\mathcal{S}_\omega$ and of the Cohen class representation Q in $\mathcal{S}_\omega$ and $\mathcal{S}'_\omega$. In particular, we give a characterization of the spaces $\mathcal{S}_\omega$, that improves a result of [11], introducing a new kind of seminorms in the spirit of the spaces of ultradifferentiable functions introduced by Braun, Meise and Taylor [8] (compare with Langenbruch [19]). See, for instance, [1,4,6] for recent results on regularity of linear partial differential operators in the local sense in this setting.

The examples that we can construct with our technique are quite general, we mention here some cases. We show for example that, if b is a polynomial in one variable that never vanishes, and $P(D_x, D_y)$ is an arbitrary partial differential operator with constant real coefficients, then the operator

$$b(x + P(D_x, D_y))$$

in $\mathbb{R}^2$ is regular in the sense of Shubin and in the sense of ultradifferentiable classes $\mathcal{S}_\omega$. The same is true for the operator in two variables

$$(x - D_y + Q(D_x))^2 + (y + R(D_y))^2,$$

for arbitrary ordinary differential operators $Q(D_x)$ and $R(D_y)$ with constant real coefficients. Observe in particular that the regularity here does not depend on the higher order terms, since the operators P, Q, R can have arbitrary order.

The paper is organized as follows. Section 2 is devoted to the study of some properties of the Wigner transform in $\mathcal{S}$, that we use in the following; in Sections 3 and 4 we study the global regularity through