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Weighted weak formulation for a nonlinear degenerate parabolic equation arising in chemotaxis or porous media

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ABSTRACT

This paper is devoted to the mathematical analysis of a degenerate nonlinear parabolic equation. This kind of equations stems either from the modeling of a compressible two phase flow in porous media or from the modeling of a chemotaxis-fluid process. In the degenerate equation, the strong nonlinearities are technically difficult to be controlled by the degenerate dissipative term because the equation itself presents degenerate terms of order 0 and of order 1. In the case of the sub-quadratic degeneracy of the dissipative term at one point, a weak and classical formulation is possible for the expected solutions. However, in the case of the degeneracy of the dissipative term at two points, we obtain solutions in a weaker sense compared to the one of the classical formulation. Therefore, a degenerate weighted formulation is introduced taking into account the degeneracy of the dissipative term.

1. Introduction and the nonlinear degenerate model

Let $T > 0$ be a fixed time and Ω be an open bounded subset of $\mathbb{R}^d$, $d = 2, 3$. We set $Q_T := \Omega \times (0, T)$ and $\Sigma_T = \partial\Omega \times (0, T)$. We consider the following nonlinear degenerate parabolic equation

$$\partial_t u - \operatorname{div}\left(a\left(u\right)\nabla u - f\left(u\right)\mathbf{V}\right) - g\left(u\right)\operatorname{div}\left(\mathbf{V}\right) + a\left(u\right)\nabla u \cdot \tilde{\mathbf{V}} = 0, \quad \text{in } Q_T. \tag{1}$$

The boundary condition is defined by

$$u\left(\mathbf{x}, t\right) = 0, \quad \text{on } \Sigma_T. \tag{2}$$

The initial condition is given by

$$u\left(\mathbf{x}, 0\right) = u_0\left(\mathbf{x}\right), \quad \text{in } \Omega. \tag{3}$$

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Models for chemotaxis lead to such kind of degenerate nonlinear parabolic equation (1), where u represents the cell density and $\mathbf{V}$ represents the gradient of the chemical concentration (see e.g. [13,1,8,16,7,17]), and in the case of swimming bacteria, $\tilde{\mathbf{V}}$ represents the velocity of the fluid which transports the cell density and the chemical concentration; in [16,7,17] the authors consider $\tilde{\mathbf{V}}$ as the Navier–Stokes velocity. In the chemotaxis modeling, the functions a and f represent respectively the diffusivity of the cells and the chemosensitivity of the cells to the chemicals. In the specific model in [13], the authors consider the case where the function a degenerates at one side and consider also a relationship between the degeneracy of the functions a and f to establish the existence and uniqueness of weak solutions. Here, we treat the case of two-sidedly degenerate diffusion terms and consider a general model.

Many physical models lead also to degenerate nonlinear parabolic problem. For instance, in [10] the authors analyzed a model of a degenerate nonlinear system arising from compressible two-phase flows in porous media. The described system coupled the saturation (denoted by u) and the global pressure (denoted by p). The global velocity (denoted by $\mathbf{V}$) is taken to be proportional to the gradient of the global pressure. In addition, the functions a and f represent respectively the capillary term and the fractional flow and the velocity $\tilde{\mathbf{V}}$ is considered to be $\tilde{\mathbf{V}} = \gamma \mathbf{V}$ where γ is a nonnegative parameter representing the compressibility factor. Several papers are devoted to the mathematical analysis of nonlinear degenerate parabolic diffusion–convection equations arising in compressible, immiscible displacement models in porous media (see e.g. [12,18]). Here, we consider a generalization of the saturation equation where we assume that the velocity field is given and fixed.

In the paper of Bresch et al. [2], the authors studied the existence of strong and weak solutions for multiphase incompressible fluids models; indeed, they consider the Kazhikhov–Smagulov system where the density equation contains a degenerate diffusion term and first order term.

In [11], the main interest is a nonlinear degenerate parabolic equation where the flux function depends explicitly on the spatial location for which they study the uniqueness and stability of entropy solutions; the studied equation do not contain first and Oth order term. The type of equation (1) arising also in sedimentation–consolidation processes [5,6,4] where the sought u is considered to be the local volume fraction of solids, many constitutive equations imply that there exists a critical number u_c such that $a(u) = 0$ for $u \le u_c$ which corresponds to the sedimentation step and $a(u) > 0$ in the consolidation step (see eq. (42) in [5]). Consequently, partial differential equations of type (1) model a wide variety of phenomena, ranging from porous media flow, via chemotaxis model, to traffic flow [20].

Our basic requirements on system (1)–(3) are:

(H1) $a \in \mathscr{C}^1([0,1], \mathbb{R})$, $a(u) > 0$ for $0 < u < 1$, $a(0) = 0$, $a(1) = 0$.
Furthermore, there exist $r_1 > 0$, $r_2 > 0$, m_1, $M_1 > 0$, and $0 < u_* < 1$ such that

$$m_1 r_1 u^{r_1 - 1} \le a'(u) \le M_1 r_1 u^{r_1 - 1}, \text{ for all } 0 \le u \le u_*,$$
$$-r_2 M_1 (1-u)^{r_2 - 1} \le a'(u) \le -r_2 m_1 (1-u)^{r_2 - 1}, \text{ for all } u_* \le u \le 1.$$

(H2) f is a differentiable function in $[0,1]$ and $g \in \mathscr{C}^1([0,1])$ verifying

$$g(0) = f(0) = 0, \quad f(1) = g(1) = 1, \text{ and } g'(u) \ge C_{g'} > 0 \quad \forall u \in [0,1].$$

In addition, there exists $c_1, c_2 > 0$ such that $|f(u) - g(u)| \le c_2 u$ for all $0 \le u \le u_*$ and $c_1 (1-u)^{-1} \le (f(u) - g(u))^{-1} \le c_2 (1-u)^{-1}$ for all $u_* \le u < 1$.

(H3) The velocities $\mathbf{V}$ and $\tilde{\mathbf{V}}$ are two measurable functions lying into $(L^\infty(\Omega))^d$.

(H4) The initial condition u_0 satisfies: $0 \le u_0(\mathbf{x}) \le 1$ for a.e. $\mathbf{x} \in \Omega$.