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Hilbert A-modules[☆]

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ABSTRACT

We consider real pre-Hilbert modules H on Archimedean f-algebras A with unit e. We provide conditions on A and H such that a Riesz representation theorem for bounded/continuous A-linear operators holds.

1. Introduction

Let A be an Archimedean f-algebra with (multiplicative) unit $e \neq 0$. It is well known that Archimedean f-algebras are commutative. We next proceed by defining the objects we study in this paper.

Definition 1. An abelian group $(H, +)$ is an *A-module* if and only if an outer product $\cdot : A \times H \to H$ is well defined with the following properties, for each $a, b \in A$ and for each $x, y \in H$:

(1) $a \cdot (x + y) = a \cdot x + a \cdot y$;
(2) $(a + b) \cdot x = a \cdot x + b \cdot x$;
(3) $a \cdot (b \cdot x) = (ab) \cdot x$;
(4) $e \cdot x = x$.

An A-module is a *pre-Hilbert A-module* if and only if an inner product $\langle\ ,\ \rangle_H : H \times H \to A$ is well defined with the following properties, for each $a \in A$ and for each $x, y, z \in H$:

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(5) $\langle x, x\rangle_H \geq 0$, with equality if and only if $x = 0$;
(6) $\langle x, y\rangle_H = \langle y, x\rangle_H$;
(7) $\langle x + y, z\rangle_H = \langle x, z\rangle_H + \langle y, z\rangle_H$;
(8) $\langle a \cdot x, y\rangle_H = a \langle x, y\rangle_H$.

For $A = \mathbb{R}$ conditions (1)–(4) define vector spaces, while (5)–(8) define pre-Hilbert spaces. We will use Latin letters a, b, c to denote elements of A, Latin letters x, y, z to denote elements of H, and Greek letters α, β to denote elements of $\mathbb{R}$.

It is well known that[1]

$$\langle x, y\rangle_H^2 \leq \langle x, x\rangle_H \langle y, y\rangle_H \qquad \forall x, y \in H.$$

We can thus conclude that each $z \in H$ induces a map $f : H \to A$, via the formula

$$f(x) = \langle x, z\rangle_H \qquad \forall x \in H,$$

with the following properties:

– **A-linearity** $f(a \cdot x + b \cdot y) = af(x) + bf(y)$ for all $a, b \in A$ and for all $x, y \in H$;
– **Boundedness** There exists $c \in A_+$ such that $f(x)^2 \leq c \langle x, x\rangle_H$ for all $x \in H$.

In light of this fact, we give the following definition:

Definition 2. Let A be an Archimedean f-algebra with unit e and H a pre-Hilbert A-module. We say that H is *self-dual* if and only if for each $f : H \to A$ which is A-linear and bounded there exists $y \in H$ such that

$$f(x) = \langle x, y\rangle_H \qquad \forall x \in H.$$

The goal of this paper is to provide conditions on A and H that will allow us to conclude that a pre-Hilbert A-module H is self-dual. Our initial motivation comes from Finance. There, Hilbert modules are the extension of the notion of Hilbert spaces that the analysis of conditional information requires, as first shown by Hansen and Richard [24]. In particular, self-duality is key to represent price operators through traded stochastic discount factors. Our results provide the general mathematical framework where conditional asset pricing can be performed.

Examples Consider a probability space $(\Omega, \mathcal{F}, P)$ and assume that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$. Denote by $\mathcal{L}^0(\mathcal{F}) = \mathcal{L}^0(\Omega, \mathcal{F}, P)$ and $\mathcal{L}^\infty(\mathcal{F}) = \mathcal{L}^\infty(\Omega, \mathcal{F}, P)$, respectively, the space of $\mathcal{F}$-measurable functions and the space of $\mathcal{F}$-measurable and essentially bounded functions. Similarly, define $\mathcal{L}^0(\mathcal{G})$ and $\mathcal{L}^\infty(\mathcal{G})$. Define also

$$\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{f \in \mathcal{L}^0(\mathcal{F}) : \mathbb{E}\left(f^2 || \mathcal{G}\right) \in \mathcal{L}^0(\mathcal{G})\right\}$$

and

$$\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P) = \left\{f \in \mathcal{L}^0(\mathcal{F}) : \mathbb{E}\left(f^2 || \mathcal{G}\right) \in \mathcal{L}^\infty(\mathcal{G})\right\}.$$

The inner product, in both cases, can be defined by $(f, g) \mapsto \mathbb{E}(fg || \mathcal{G})$. In Section 6, we show that $\mathcal{L}^{2,0}(\Omega, \mathcal{G}, \mathcal{F}, P)$ is a pre-Hilbert $\mathcal{L}^0(\mathcal{G})$-module and $\mathcal{L}^{2,\infty}(\Omega, \mathcal{G}, \mathcal{F}, P)$ is a pre-Hilbert $\mathcal{L}^\infty(\mathcal{G})$-module. Both

[1] See Huijsmans and de Pagter [28, Theorem 3.4] and also Proposition 4 below.