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Existence and extendibility of rotationally symmetric graphs with a prescribed higher mean curvature function in Euclidean and Minkowski spaces ☆

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ABSTRACT

In this paper we investigate the existence of rotationally symmetric entire graphs (resp. entire spacelike graphs) with prescribed k-th mean curvature function in Euclidean space $\mathbb{R}^{n+1}$ (resp. Minkowski spacetime $\mathbb{L}^{n+1}$). As a previous step, we analyze the associated homogeneous Dirichlet problem on a ball, which is not elliptic for $k > 1$, and then we prove that it is possible to extend the solutions. Moreover, a sufficient condition for uniqueness is given in both cases.

1. Introduction

Along this paper, $\mathbb{R}^{n+1}_a$, $a = 0, 1$, will denote, for $a = 0$, the $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ endowed with its standard Riemannian metric $\langle\, ,\, \rangle = \sum_{i=1}^{n+1} dx_i^2$ and, for $a = 1$, the $(n+1)$-dimensional Lorentzian spacetime $\mathbb{L}^{n+1}$ endowed with its standard Lorentzian metric $\langle\, ,\, \rangle = -dx_1^2 + \sum_{j=2}^{n+1} dx_j^2$ and with the time orientation defined by $\partial/\partial x_1$. For a two sided hypersurface ($a = 0$) or spacelike hypersurface ($a = 1$) in $\mathbb{R}^{n+1}_a$, the k-th mean curvatures are geometric invariants which encode the geometry of the hypersurface. From an algebraic point of view, each one of these functions corresponds to a coefficient of the characteristic polynomial of the shape operator corresponding to a unit normal vector field ($a = 0$) or to a unit timelike vector field pointing to future ($a = 1$). In fact, each k-th mean curvature is described as a certain type of average measure of the principal curvatures of the hypersurface (see Section 2 for details). In particular, the 1-th mean curvature corresponds with the usual mean of the principal curvatures if $a = 0$ or its opposite if $a = 1$, the 2-th mean curvature is, up to a constant factor, the scalar curvature, and the

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n-th mean curvature is the Gauss–Kronecker curvature if $a = 0$ and $(-1)^{n+1}$ times the Gauss–Kronecker curvature if $a = 1$. Each k-th mean curvature has a variational nature [21], and in Riemannian geometry the constant k-th mean curvature case has been extensively studied ([16,22] for instance). From a physical perspective, the k-th mean curvatures have a relevant role in General Relativity. A spacelike hypersurface is a suitable subset of the spacetime where the initial value problems for the different field equations are naturally stated. Roughly speaking, a spacelike hypersurface represents the physical space at one time instant. Each k-th mean curvature function intuitively measures the time evolution towards the future or the past of the spatial universe (see Remark 2.1).

In the Euclidean context, the pioneer work on the prescribed mean curvature Dirichlet problem was given by Serrin [23], who found necessary and sufficient conditions for its solvability. In Minkowski spacetime, Cheng and Yau proved in [9] the Bernstein's property for entire solutions of the maximal (i.e., zero mean curvature) hypersurface equation and, later, Treibergs [25] classified the entire solutions of the constant mean curvature spacelike hypersurface equation. An important universal existence result was proved by Bartnik and Simon [3], and Bartnik proved the existence of prescribed mean curvature spacelike hypersurfaces under certain asymptotic assumptions [2]. The Dirichlet problem in a more general spacetime was solved by Gerhardt [14]. More recently, there are more contributions (see for instance, [1]) and the interest is many times focused on the existence of positive solutions, by using a combination of variational techniques, critical point theory, sub-supersolutions and topological degree (see for instance [6,7,10–12] and the references therein). Respect to the scalar curvature, we refer to [8] in the Euclidean context. On the other hand, Bayard proved the existence of prescribed scalar entire spacelike hypersurfaces in Minkowski spacetime [4], by using other previous works on the Dirichlet problem ([5] and [26] and references therein) and Gerhardt [15] obtained important results on the case of more general ambient spacetimes. Finally, the Gauss–Kronecker curvature has been also quite well studied in both settings. In Euclidean space, Wang [28] prescribed the Gauss–Kronecker curvature of a convex hypersurface. In Minkowski spacetime, we highlight the work of Li [19] on constant Gauss curvature and Delanoè [13], in which the existence of entire spacelike hypersurfaces asymptotic to a lightcone with prescribed Gauss–Kronecker curvature function is proved.

Up to the last decade, little attention has been paid to hypersurfaces with prescribed k-th mean curvature when $3 \leq k < n$. One of the first works in this direction was done by Ivochkina (see [18] and references therein). More recently, several contributions (for instance, [27,17]) especially on the Dirichlet problem has been done. However, the general question is still open in both settings. The study has usually been focused in the search of some *a priori* bounds on the length of the shape operator, assuming that solutions are k-stable to ensure the ellipticity of some involved differential operators (see [27] for more details). Then, some special dependence in the prescription function is imposed in order to obtain partial results. In this paper, we provide several existence and uniqueness results on this open problem, assuming that the prescription function is rotationally symmetric respect to a unit parametrized line or an inertial observer γ (i.e., a unit timelike parametrized line pointing to the future) in $\mathbb{R}^{n+1}_a$ with $a = 0$ or $a = 1$, respectively. Especially, we prove the existence of rotationally symmetric entire graphs with prescribed k-th mean curvature of the associated Dirichlet problem when the domain in a n-dimensional ball, by using a suitable fixed point operator. Besides, we prove that such graph can be extended to the whole space, providing some information about uniqueness as well.

Along the paper, we will use the usual cylindrical coordinates (t, r, Θ) in $\mathbb{R}^{n+1}_a$ associated to γ, namely, $t \in \mathbb{R}$ is the parameter of γ, $r \in \mathbb{R}^+$ is the radial distance to γ and $\Theta = (\theta_1, ..., \theta_{n-1})$ are the standard spherical coordinates of the $(n-1)$-dimensional unit round sphere $\mathbb{S}^{n-1}$. The prescription functions H_k will be assumed to be radially symmetric with respect to γ. Therefore, it is natural to consider $H_k(t, x) = H_k(t, r)$ where r denotes the distance of $x \in \mathbb{R}^n$ to γ.

Hypersurfaces in Euclidean space and spacelike hypersurface in Lorentz–Minkowski spacetime have a different geometry. Therefore, in the related literature, one can make a clear distinction between two large groups of papers, depending if they consider the Euclidean or the Lorentzian ambient. However, we have