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Eliminating unphysical photon components from Dirac–Maxwell Hamiltonian quantized in the Lorenz gauge

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A R T I C L E I N F O

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A B S T R A C T

We study the Dirac–Maxwell model quantized in the Lorenz gauge. In this gauge, the space of quantum mechanical state vectors inevitably adopt an indefinite metric so that the canonical commutation relation (CCR) is realized in a Lorentz covariant manner. In order to obtain a physical subspace, in which no negative norm state exists, the method first proposed by Gupta and Bleuler is applied with mathematical rigor. It is proved that a suitably defined physical subspace has a positive semi-definite metric, and naturally induces a physical Hilbert space with a positive definite metric. Then, the original Dirac–Maxwell Hamiltonian defines an induced Hamiltonian on the physical Hilbert space which is essentially self-adjoint.

1. Introduction

We consider a quantum system of N Dirac particles under an external potential V interacting with a quantized gauge field (so called Dirac–Maxwell model). If we apply the informal perturbation theory to this model, quantitative predictions are obtained such as the Klein–Nishina formula for the cross section of the Compton scattering of an electron and a photon [19], which agrees with the experimental results very well. Hence, the Dirac–Maxwell model is expected to describe a certain class of realistic quantum phenomena and thus is worth the analysis with mathematical rigor, even though it may suffer from so called "negative energy problem". The mathematically rigorous study of this model was initiated by Arai in Ref. [1], and several mathematical aspects of the model was analyzed so far (see, e.g., Refs. [3–5,22], and [23]).

The motivation of the present study is to treat this model in the Lorentz gauge in which the Lorenz covariance is manifest. In analyzing gauge theories such as quantum electrodynamics (QED) in a Lorentz covariant gauge, a difficulty always arises, since one must adopt "an indefinite metric Hilbert space" as a space of all the state vectors (for instance, see Ref. [11]). In such cases, we have to pick up a positive definite subspace from the total space and regard it as the subspace consisting of the physical state vectors.

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This is done by eliminating unphysical photon modes with negative norms. The most general and elegant method to identify the physical subspace for (possibly non-abelian) gauge theories quantized in a covariant gauge was given by the celebrated work by Kugo and Ojima [13,14], which is based on the BRST symmetry, the remnant gauge symmetry of the Lagrangian density after imposing some gauge fixing condition. The Kugo–Ojima formulation reduces to Nakanishi and Lautrup's B-field theory [15–18] in the case where the gauge field is abelian and after integrating out the auxiliary Nakanishi–Lautrup's B-field, it is reduced to the condition first proposed by Gupta and Bleuler [6,9].

The Gupta and Bleuler condition says that a state vector belongs to the physical subspace if and only if it has the vanishing expectation value of the operator $\partial_\mu A^\mu$, the four component divergence of the gauge field:

$$\langle \Psi | \partial_\mu A^\mu(t, \boldsymbol{x}) \Psi \rangle = 0, \tag{1.1}$$

at every spacetime point $(t, \boldsymbol{x}) \in \mathbb{R}^4$. From the equations of motion $\Box A_\mu = -j_\mu$ (with "mostly plus metric") and the current conservation equation $\partial_\mu j^\mu = 0$, we heuristically find that $\partial_\mu A^\mu(t, \boldsymbol{x})$ satisfies the Klein–Gordon equation so that (1.1) is written as

$$[\partial_\mu A^\mu]^+(t, \boldsymbol{x})\Psi = 0, \tag{1.2}$$

where $[\partial_\mu A^\mu]^+(t, \boldsymbol{x})$ denotes the positive frequency part of the free field $\partial_\mu A^\mu$. However, in order to rigorously follow this procedure, one has to answer the following two questions. Firstly, how to identify $A(t, \boldsymbol{x})$ at a time $t \in \mathbb{R}$? Since the present state vector space is not an ordinary Hilbert space with a positive definite metric, the Hamiltonian can not be defined as a self-adjoint operator in the ordinary sense. Thus, it is far from trivial if there is a solution of quantum Heisenberg equations of motion. Secondly, is it possible to identify the positive frequency part of the operator satisfying Klein–Gordon equation even in an indefinite metric space? The first problem is solved by the general construction method of time evolution operator generated by a non-self-adjoint operator given by the authors [7], and as to the second one, the general definition of "positive frequency part" of a quantum field satisfying Klein–Gordon equation is given in Ref. [10].

Mathematically rigorous study of concrete models of QED in the Lorentz covariant gauge (see, for instance, Refs. [10,12,24,26]) was only given for a solvable models as far as we know. However, the model treated here, the Dirac–Maxwell model, is *not* solvable in the sense that an explicit expression of the time-dependent gauge field is not easily found. Thus, the problem has to be abstractly considered without relying on the explicit expression of the time dependent gauge field but only on the abstract existence theorem. In this paper, we establish the existence of a time evolution operator and give a definition of the "positive frequency part" of a free field satisfying Klein–Gordon equation in an abstract setup. Our definition of "positive frequency part" given here is different from that given in Ref. [10], but results in the same consequence when applied to the concrete models. We then apply to the abstract theory to the concrete Dirac–Maxwell model and identify the physical Hilbert space. We also prove that the original Dirac–Maxwell Hamiltonian naturally defines a self-adjoint "physical Hamiltoninan" on the Hilbert space, which is essentially equivalent to the Dirac–Maxwell Hamiltonian in the Coulomb gauge discussed in Ref. [1].

The mathematical tools developed here would have some interests in its own right. The time evolution operator generated by *unbounded, non-self-adjoint* Hamiltonians has been constructed in Ref. [7]. In this paper, we further develop the theory in several aspects. Firstly, we define a general class of operators, which we will call $\mathcal{C}_n$-class operators, and prove that a $\mathcal{C}_n$-class operator B has a time evolution $B(t)$ for $t \in \mathbb{R}$ which solves the Heisenberg equation, and $B(t)\xi$ is n-times strongly differentiable in t for ξ belonging to a dense subset D'. Moreover, the n-th derivative of $B(t)$ enjoys the natural expression in terms of a weak commutator defined in a suitable sense. Secondly, we define a more restricted class of operators, which will be called $\mathcal{C}_\omega$-class operators, and prove that $\mathcal{C}_\omega$-class operators have a time evolution which is analytic in $t \in \mathbb{R}$. Furthermore, it would be interesting to see that the following Taylor expansion formula (2.40) remains valid