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Spectral analysis for stability of bubble steady states of a Keller–Segel's minimal chemotaxis model ☆

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ABSTRACT

A number of techniques, some of which are novel, are introduced to develop a systematic method to study a set of eigenvalue problems arising from the stability analysis of bubble steady states of a Keller–Segel's minimal chemotaxis model. Estimates of the eigenvalue with largest real part of an elliptic system without variational structure and the second eigenvalue of a corresponding subproblem possessing variational structure are obtained. These estimates provide critical information about the stability of the bubble steady state with respect to the time relaxation parameter; in particular, it is shown that the stability decreases to zero as the relaxation parameter goes to infinity.

1. Introduction

Spectral analysis is extremely important in physics and engineering. Estimation of size of eigenvalues provides valuable information about the observability and duration of observation of special phenomena and about the effects of small perturbation from equilibria. In this paper we consider eigenvalue problems arising from a Keller and Segel's [10] *minimal chemotaxis model*, which in its dimensionless form, can be written as

$$\begin{cases} \tau u_t = (u_x - kf(u)v_x)_x & \text{in } \Omega \times (0,\infty), \\ v_t = v_{xx} - v + g(u) & \text{in } \Omega \times (0,\infty), \\ u_x = v_x = 0 & \text{on } \partial\Omega \times (0,\infty), \\ \int_\Omega u(x,t)dx = m & \text{for all } t \geq 0; \end{cases} \tag{1.1}$$

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here $\Omega = (0, \ell)$, t relates the time, x the space, u the cell density, v the chemo-attractant density, m the total mass of the cell, f and g are given functions to be specified later, and $\tau \geqslant 0$ and $k \gg 1$ are constants. We call τ the **time relaxation parameter** and k the chemotaxis' strength parameter.

Any steady state of (1.1) can be extended evenly and periodically. We assume that ℓ is the half-period. Then steady states are solutions of

$$\begin{cases} u_x = kf(u)v_x > 0 & \text{in } \Omega, \\ v_{xx} - v + g(u) = 0 & \text{in } \Omega, \\ u_x = v_x = 0 & \text{on } \partial\Omega, \\ \int_\Omega u(x) = m. \end{cases} \tag{1.2}$$

A huge volume of work has been devoted to various PDE models for chemotaxis; see the survey papers [5–7]. The most important phenomenon about chemotaxis is cell aggregation, which typically is modeled by spiky steady states. The pioneering papers that prove the existence of such steady states are Lin, Ni and Takagi [12,13], Kabeya and Ni [8]; see also Wang and Xu [15] for a brief survey.

In [15], Wang and Xu proved the following: (i) Suppose $f(u) = u$ and $g(u) = u$. Then (1.1) admits a steady state solution of which u approaches, as $k \to \infty$, a Dirac measure of mass m, for which we call a *spike*; (ii) Suppose $f(u) = u - u^2$, $g(u) = u$ and $m \in (0, \ell)$. Then (1.1) admits a steady state solution of which u approaches, as $k \to \infty$, a translated Heaviside function, for which we call a *bubble*.

For spike solutions, Kang, Kolokolnikov, and Ward [9] derived formally the first term of the expansion (as $k \to \infty$). In [2], we established the uniqueness of the spike solution, provided its rigorous asymptotic expansion, and proved that the solution is locally exponentially stable. The associated eigenvalue problem is reinvestigated in [16] in a general setting with a systematic method.

Very recently, in [11] we extended the analysis of [2] to the bubble solution, establishing the existence, uniqueness, asymptotic expansion, and local exponentially stability of the bubble solution discovered by Wang and Xu [15], under the following structural conditions:

(f) $f \in C^3([0,1])$, $f(0) = f(1) = 0$, $f > 0$ in $(0,1)$, $f'(0) > 0 > f'(1)$;
(g) $g \in C^2([0,1])$, $g(0) = 0$, $g(1) = 1$, $g' > 0$ in $(0,1)$.

The stability of a steady state (u, v) of (1.1) is determined by the principal eigenvalue (that with largest real part) of the corresponding eigenvalue problem

$$\begin{cases} \tau\lambda\phi = (\phi_x - kf(u)\psi_x - kf'(u)v_x\phi)_x & \text{in } \Omega, \\ \lambda\psi = \psi_{xx} - \psi + g'(u)\phi & \text{in } \Omega, \\ \phi_x = \psi_x = 0 & \text{on } \partial\Omega, \\ \int_\Omega \phi dx = 0. \end{cases} \tag{1.3}$$

Set $p = kf(u)$ and $w = \phi/p - \psi$, so $\phi = p[w + \psi]$. Using $u_x = pv_x$, one can show that (1.3) is equivalent to

$$\begin{cases} (pw_x)_x = \tau\lambda p[w + \psi] & \text{in } \Omega, \\ \psi_{xx} - \psi + g'p[w + \psi] = \lambda\psi & \text{in } \Omega, \\ w_x = \psi_x = 0 & \text{on } \partial\Omega, \\ \int_\Omega p[w + \psi]dx = 0. \end{cases} \tag{1.4}$$

For bubble solutions, in [11] we proved that when $k \gg 1$,