J. Math. Anal. Appl. 446 (2017) 1133–1138

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Note

Blow-up for a thin-film equation with positive initial energy ☆

Jun Zhou

School of Mathematics and Statistics, Southwest University, Chongqing, 400715, PR China

A R T I C L E   I N F O

Article history:
Received 16 July 2016
Available online 14 September 2016
Submitted by J. Shi

Keywords:
Thin-film equation
Positive initial energy
Blow-up
Upper bound of blow-up time

A B S T R A C T

In this paper, we consider a thin-film equation with nonlocal source, which was studied by Qu and Zhou (2016) [11], where the authors derived the conditions for global existence, blow-up and extinction. We consider the case that the initial energy is positive and establish a blow-up result for this case. Furthermore, the upper bound of the blow-up time is derived.

1. Introduction

Recently, the evolution equations with nonlocal source like $|u|^{p-1}u - |\Omega|^{-1}\int_\Omega |u|^{p-1}udx$ were studied extensively (see [1,2,4–12] and references therein). Particularly, in [11], Qu and Zhou considered the following thin-film equation:

$$\begin{cases} u_t + u_{xxxx} = |u|^{p-1}u - \displaystyle\fint_0^a |u|^{p-1}udx, & x \in (0,a),\ t > 0, \\ u_x(0) = u_x(a) = u_{xxx}(0) = u_{xxx}(a) = 0, & t > 0, \\ u(x,0) = u_0(x), & x \in (0,a), \end{cases} \tag{1.1}$$

where a is a positive constant, $p > 1$ and $u_0 \in H^2(0,a)$, $\fint_0^a u_0 dx = a^{-1}\int_0^a u_0 dx$ with $u_0 \not\equiv 0$.

In [11], the authors considered the global existence, blow-up and extinction of the solutions to (1.1). Now, we recall some notations and functionals in that paper. Denote by $\|\cdot\|_q$ the $L^q(0,a)$ norm for $1 \le q \le \infty$ and define

☆ This work is partially supported by the Natural Science Foundation of Chongqing cstc2016jcyjA0018, Fundamental Research Funds for the Central Universities grant XDJK2015A16, XDJK2016E120, Project funded by China Postdoctoral Science Foundation grant 2014M550453, 2015T80948.

E-mail address: jzhouwm@163.com.

http://dx.doi.org/10.1016/j.jmaa.2016.09.026
0022-247X/

$$J(u) = \frac{1}{2}\|u_{xx}\|_2^2 - \frac{1}{p+1}\|u\|_{p+1}^{p+1}. \tag{1.2}$$

A natural question is what is the lifenspan (upper bound of the blow-up time) to the blow-up solutions. In this paper, under the condition $J(u_0) > 0$, we will establish a new blow-up condition and estimate the upper bound of the blow-up time.

Let $u(x,t)$ be the solution of problem (1.1), we can easily find that $u(x,t)$ satisfies $\int_0^a u_t dx = 0$, which further implies that $\int_0^a u dx = \int_0^a u_0 dx = 0$.

Let $\mathcal{W} \triangleq \left\{u \in H^2(0,a) : \int_0^a u dx = 0\right\}$, then $(\mathcal{W}, \|\cdot\|)$ with $\|u\| \triangleq \|u_{xx}\|_2$ is a Banach space. By using the Sobolev Embedding Theorem [3], we know that $\mathcal{W} \hookrightarrow L^{p+1}(0,a)$ continuously. Let B be the optimal constant of the embedding, i.e.,

$$\|u\|_{p+1} \leq B\|u_{xx}\|_2. \tag{1.3}$$

Let α_1 and E_1 be two positive constants defined as follows:

$$\begin{aligned} &\alpha_1 \triangleq B^{-\frac{p+1}{p-1}}, \\ &E_1 \triangleq \frac{p-1}{2(p+1)} B^{-\frac{2(p+1)}{p-1}} = \frac{p-1}{2(p+1)}\alpha_1^2. \end{aligned} \tag{1.4}$$

With the notations given above, the main results of this paper can be stated as follows:

Theorem 1.1. *Assume $0 < J(u_0) < E_1$ and $\|u_{0xx}\|_2 > \alpha_1$, then the solution $u(x,t)$ to problem (1.1) blows up at a finite time T_*. Moreover, T_* can be estimate by*

$$T_* \leq \frac{(p+1)a^{\frac{p-1}{2}}\|u_0\|_2^{-(p-1)}}{(p-1)^2\left\{1 - \left[(p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right)\right]^{-\frac{p+1}{p-1}}\right\}}. \tag{1.5}$$

Remark 1.2. By (1.4) and the fact that $J(u_0) < E_1$, we know that

$$\begin{aligned} (p+1)\left(\frac{1}{2} - \frac{J(u_0)}{\alpha_1^2}\right) &> (p+1)\left(\frac{1}{2} - \frac{E_1}{\alpha_1^2}\right) \\ &= (p+1)\left(\frac{1}{2} - \frac{p-1}{2(p+1)}\right) = 1, \end{aligned}$$

then it is easy to see the denominator of the right-hand side of (1.5) is positive, which means (1.5) makes sense.

2. Proof of Theorem 1.1

To prove Theorem 1.1, we need to introduce several lemmas. The first Lemma can be got by [11, (1.6)].

Lemma 2.1. *The $J(u)(t)$ defined in (1.2) is nonincreasing in t and*

$$J(u) = J(u_0) - \int_0^t \|u_\tau\|_2^2 d\tau.$$