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Glazman–Krein–Naimark theory, left-definite theory and the square of the Legendre polynomials differential operator

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A R T I C L E I N F O

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We dedicate this paper to the memory of W.N. (Norrie) Everitt (1924–2011)

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A B S T R A C T

As an application of a general left-definite spectral theory, Everitt, Littlejohn and Wellman, in 2002, developed the left-definite theory associated with the classical Legendre self-adjoint second-order differential operator A in $L^2(-1, 1)$ which has the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ as eigenfunctions. As a consequence, they explicitly determined the domain $\mathcal{D}(A^2)$ of the self-adjoint operator A^2. However, this domain, in their characterization, does not contain boundary conditions. In fact, this is a general feature of the left-definite approach developed by Littlejohn and Wellman. Yet, the square of the second-order Legendre expression is in the limit-4 case at each end point $x = \pm 1$ in $L^2(-1, 1)$ so $\mathcal{D}(A^2)$ should exhibit four boundary conditions. In this paper, we show that this domain can, in fact, be expressed using four separated boundary conditions using the classical GKN (Glazman–Krein–Naimark) theory. In addition, we determine a new characterization of $\mathcal{D}(A^2)$ that involves four *non-GKN* boundary conditions. These new boundary conditions are surprisingly simple – and natural – and are equivalent to the boundary conditions obtained from the GKN theory.

1. Introduction

The analytical study of the classical second-order Legendre differential expression

$$\ell[y](x) = -\left((1 - x^2)y'(x)\right)'$$

has a long and rich history stretching back to the seminal work of H. Weyl in 1910 [23] and E.C. Titchmarsh in 1940 [21]. Part, if not most, of the reason for the importance of this second-order expression lies in the fact that the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are solutions. More specifically, the Legendre polynomial $y = P_n(x)$, for $n \in \mathbb{N}_0$, is a solution of the eigenvalue equation

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$$\ell[y](x) = n(n+1)y(x).$$

In the Hilbert space $L^2(-1,1)$, there is a continuum of self-adjoint operators generated by $\ell[\cdot]$. One such operator A stands out from the rest: this is the Legendre polynomials operator, so named because the Legendre polynomials $\{P_n\}_{n=0}^{\infty}$ are eigenfunctions of A. We review properties of this operator in Section 2.

In the mid 1970s, Å. Pleijel wrote two papers (see [18] and [19]) on the Legendre expression from a left-definite spectral point of view. W.N. Everitt's contribution [8] continued this left-definite study in addition to detailing an in-depth analysis of the Legendre expression in the right-definite setting $L^2(-1,1)$ where he discovered new properties of functions in the domain $\mathcal{D}(A)$ of A. In [14], A.M. Krall and Littlejohn considered properties of the Legendre expression under the left-definite energy norm. In 2000, R. Vonhoff extended Everitt's results in [22] with an extensive study of $\ell[\cdot]$ in its (first) left-definite setting. In 2002, Everitt, Littlejohn and Marić [10] published further results in which they gave several equivalent conditions for functions to belong to $\mathcal{D}(A)$; this result is given below in Theorem 1. We also refer the reader to the paper [16] by Littlejohn and Zettl where the authors determine all self-adjoint operators, generated by the Legendre expression $\ell[\cdot]$, in the Hilbert spaces $L^2(-1,1)$, $L^2(-\infty,-1)$, $L^2(1,\infty)$ and $L^2(\mathbb{R})$. At this point, we also reference the excellent text [25] by Zettl on Sturm–Liouville theory.

Littlejohn and Wellman [15], in 2002, developed a general left-definite theory for an unbounded self-adjoint operator T bounded below by a positive constant in a Hilbert space $H = (V, (\cdot,\cdot))$, where V denotes the underlying (algebraic) vector space and H is the resulting topological space induced by the norm $\|\cdot\|$ and inner product $(\cdot,\cdot)$. In a nutshell, the authors construct a continuum of Hilbert spaces $\{H_r = (V_r, (\cdot,\cdot)_r)\}_{r>0}$, forming a Hilbert scale, generated by positive powers of T. The authors called these Hilbert spaces *left-definite spaces*; they are constructed using the Hilbert space spectral theorem (see [20]) for self-adjoint operators.

It is a difficult problem, in general, to explicitly determine the domain of a power of an unbounded operator. However, the authors in [15] prove that, for $r > 0$, $V_r = \mathcal{D}(T^{r/2})$ and $(f,g)_r = (T^{r/2}f, T^{r/2}g)$. Furthermore, in many practical applications, as the authors demonstrate in [15], the computation of the vector spaces V_r and inner products $(\cdot,\cdot)_r$ is surprisingly not difficult when $r \in \mathbb{N}$. In a subsequent paper, Everitt, Littlejohn and Wellman [11] applied this theory to the Legendre polynomials operator A. Among other results, the authors explicitly compute the domains of $\mathcal{D}(A^{n/2})$ for each $n \in \mathbb{N}$. Specifically, they proved

$$\mathcal{D}(A^{n/2}) = \{f : (-1,1) \to \mathbb{C} \mid f, f', \ldots, f^{(n-1)} \in AC_{\mathrm{loc}}(-1,1); (1-x^2)^{n/2} f^{(n)} \in L^2(-1,1)\} \quad (n \in \mathbb{N}). \tag{1.1}$$

In particular, we see that $\mathcal{D}(A^2)$ is explicitly given by

$$B = \{f : (-1,1) \to \mathbb{C} \mid f, f', f'', f''' \in AC_{\mathrm{loc}}(-1,1); (1-x^2)^2 f^{(4)} \in L^2(-1,1)\}; \tag{1.2}$$

the reason for using the notation B, instead of $\mathcal{D}(A^2)$, will be made clear shortly. Of course, for $f \in B$, we have $A^2 f = \ell^2[f]$, where $\ell^2[\cdot]$ is the square of the Legendre differential expression given by

$$\ell^2[y](x) = \left((1-x^2)^2 y''(x)\right)'' - 2\left((1-x^2)y'(x)\right)'. \tag{1.3}$$

Notice that, curiously, there are no 'boundary conditions' given in (1.2). From the Glazman–Krein–Naimark (GKN) theory [17, Theorem 4, Section 18.1], there should be *four* such boundary conditions. This begs an obvious question: how can we 'extract' boundary conditions from the representation of $\mathcal{D}(A^2)$ in (1.2)? In this paper, we will answer this question. It is interesting that the condition $(1-x^2)^2 f^{(4)} \in L^2(-1,1)$ seems to 'encode' these boundary conditions. In fact, along the way, we will characterize $\mathcal{D}(A^2)$ in four different ways. Of course, we have the algebraic definition