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Stationary solutions of continuous and discontinuous neural field equations

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ABSTRACT

We study existence and continuous dependence of the solutions to the Hammerstein operator equation under the transition from continuous nonlinearities in the Hammerstein operator to the Heaviside nonlinearity in a vicinity of the solution, corresponding to the discontinuous nonlinearity case. We apply these results to corresponding problems arising in the neural activity modeling.

1. Introduction

We consider a special case of nonlinear operator equation with the Hammerstein operator, the nonlinear part of is either represented by the Heaviside unit step function, or by a bounded continuous function. We are studying existence and continuous dependence of the solutions to the Hammerstein operator equation under the transition from continuous nonlinearities in the Hammerstein operator to the Heaviside nonlinearity. To do this, we choose an appropriate topology, where the Hammerstein operator with the Heaviside nonlinearity becomes continuous in a vicinity of the solution, corresponding to the case of the discontinuous Hammerstein operator nonlinearity. Then we use methods of functional analysis and topological degree theory to establish the results needed. This study is strongly motivated by applications of some problems arising in the neural activity modeling. Below we give a detailed descriptions of these problems.

It is well-known (see e.g. [11,9]) that electrical activity in the neocortex is naturally studied in the framework of cortical networks. However, since the number of neurons and synapses in even a small piece of cortex is immense, a suitable modeling approach is to take a continuum limit of the neural networks and, thus, consider so-called neural field models of the brain cortex (rigorous justification of this limit procedure

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can be found in e.g. [4]). The simplest model describing the macro-level neural field dynamics is the Amari model [1]

$$\partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \omega(x-y) f(u(t,y))dy, \quad t \geq 0,\, x \in \Xi \subseteq R^m. \tag{1}$$

Here $u(t,x)$ denotes the activity of a neural element u at time t and position x. The connectivity function ω determines the coupling strength between the elements and the non-negative function $f(u)$ gives the firing rate of a neuron with activity u. Neurons at a position x and time t are said to be active if $f(u(t,x)) > 0$. Typically f is a smooth function that has sigmoidal shape. Solvability of (1) in the case of a smooth firing rate function was proved in [23,3]. Particular attention in the neural field theory is usually given to the localized stationary, i.e., time-independent, solutions to (1) (so-called "bump solutions", or simply "bumps"), as they correspond to normal brain functioning (see e.g. [26]). Faugeras et al. [8] proved existence and uniqueness of the stationary solution to (1) as well as obtained conditions for this solution to be absolutely stable, for the case of a bounded Ξ.

A common simplification of (1) consists of replacing a smooth firing rate function by the Heaviside function. This replacement simplifies numerical investigations of the model as well as allows to obtain closed form expressions for some important types of solutions (see e.g. [1,22,17]). Existence of the solution to (1) in the case of Heaviside firing rate function was proved by Potthast et al. [23]. Stability of the stationary solutions to (1) is usually assessed by the Evans function approach (see e.g. [6,22]). The analysis of existence and stability of localized stationary solutions for a special class of the firing rate functions, the functions that are "squeezed" between two unit step functions, was carried out in [13,21,15]. This analysis served as a connection between stability\instability properties of the solutions to the models with the "squeezing" Heaviside firing rate functions and the solution to the model with the "squeezed" smooth firing rate function. However, no rigorous mathematical justification of the passage from a smooth to discontinuous firing rate functions in the framework of neural field models was given until the work by Oleynik et al. [20], where continuous dependence of the 1-bump stationary solution to (1) under the transition from a smooth firing rate function to the Heaviside function was proved in the 1-D case.

On the other hand, more advanced neural field models have not been studied in this respect. One example is the homogenized Amari model describing the neural field dynamics on both macro- and micro-levels

$$\begin{gathered} \partial_t u(t,x,x_{\mathrm{f}}) = -u(t,x,x_{\mathrm{f}}) + \int\limits_{\Xi}\int\limits_{\mathcal{Y}} \omega(x-y, x_{\mathrm{f}}-y_{\mathrm{f}}) f(u(t,y,y_{\mathrm{f}}))dy_{\mathrm{f}}dy, \\ t \geq 0,\ x \in \Xi,\ x_{\mathrm{f}} \in \mathcal{Y} \subset R^k, \end{gathered} \tag{2}$$

which was introduced in the pioneering work by Coombes et al. [7]. Here x_{f} is the fine-scale spatial variable and $\mathcal{Y}$ is an elementary domain of periodicity in R^k. As it was shown in [24], the solution to (2) is a weak two-scale limit of solutions to the following family of heterogeneous neural field models

$$\begin{gathered} \partial_t u(t,x) = -u(t,x) + \int\limits_{\Xi} \omega^{\varepsilon}(x-y) f(u(t,y))dy, \\ \omega^{\varepsilon}(x) = \omega(x, x/\varepsilon),\ 0 < \varepsilon \ll 1, \\ t \geq 0,\ x \in \Xi, \end{gathered} \tag{3}$$

as $\varepsilon \to 0$, where ε corresponds to the medium heterogeneity.

The starting point for the investigation of the solutions to (2) was assuming these solutions to be independent of the fine-scale variable, i.e. solutions to the equation