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Compactness and the fixed point property in ℓ_1 ☆

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ABSTRACT

In this paper we prove that compactness can be characterized by means of the existence of a fixed point for some classes of mappings defined on convex closed subsets of the space ℓ_1. Nominally, our result involves nonexpansive mappings, uniformly Lipschitzian mappings and cascading nonexpansive mappings. We also extend the results to some more general classes of Banach spaces.

1. Introduction

The main goal of this paper is to characterize compactness of a convex closed subset of ℓ_1 by means of the existence of a fixed point for several classes of mappings. Many different topological, metric or geometrical properties are often used in Fixed Point Theory to prove the existence of a fixed point. In some rare cases, these properties are not only sufficient but also necessary and so, they are characterized by fixed point results. For instance, while Schauder's Theorem assures the existence of a fixed point for continuous mappings defined on a convex compact subset C of a linear normed space X, V. Klee [18] proved that compactness is also a necessary assumption, and so, he stated: A convex closed subset C of a linear normed space is compact if and only if every continuous mapping defined from C into C has a fixed point. P.K. Lin and Y. Sternfeld [24] improved Klee's result in 1985 proving that for any convex closed noncompact subset C of a linear normed space there exists a Lipschitzian mapping f which is fixed point free. (In fact, they proved the following much stronger result: $\inf\{\|x - fx\| : x \in C\} > 0$.) Since the mapping $\lambda f + (1-\lambda)I$ ($\lambda \in (0, 1)$) has the same fixed point set as f and it is $\lambda L + (1-\lambda)$-Lipschitzian whenever f is L-Lipschitzian, letting $\lambda \to 0^+$, we can easily check that the fixed point free mapping f can be chosen with a Lipschitz constant as close to 1^+ as wanted. (For more aspects concerning the failure of Schauder's Theorem in noncompact

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setting, see [14].) To fix the notation, we will say that C satisfies the Fixed Point Property (FPP) for a class of mappings $\mathcal{A}$ if every mapping $f \in \mathcal{A}$ defined from C into C has a fixed point. Thus, the result in [24] can be stated as follows:

Theorem 1.1. *Let L be any number greater that 1. A convex closed subset C of a Banach space satisfies the FPP for L-Lipschitzian mappings if and only if it is a compact set.*

One could wonder if the same is true for $L = 1$, but it was already known that the behavior of 1-Lipschitzian mappings (i.e. nonexpansive mappings) with respect to the FPP is quite different. Indeed, F. Browder [4] had proved in 1965 that any closed convex bounded subset of a Hilbert space satisfies the FPP for nonexpansive mappings. In fact, since W. Ray [28] proved in 1985 that every closed convex unbounded set of a Hilbert space fails the FPP for nonexpansive mappings, we have the following:

Theorem 1.2. *A convex closed subset C of a Hilbert space satisfies the FPP for nonexpansive mappings if and only if it is bounded.*

The reflexivity of Hilbert spaces let us state Theorem 1.2 in the following equivalent form:

Theorem 1.3. *A convex closed subset C of a Hilbert space satisfies the FPP for nonexpansive mappings if and only if it is weakly compact.*

No similar characterization is known for any other reflexive space. Furthermore, Theorem 1.2 does not hold for the classic nonreflexive spaces c_0 and ℓ_1 because it is well known that in these spaces there are some closed convex and bounded subsets which fail the FPP for nonexpansive mappings (for instance, the closed unit ball of c_0 or the positive face of the unit sphere of ℓ_1). However, Theorem 1.3 does also hold for the space c_0 as proved in [7], extending [11] and [26]. Thus, it is natural to consider the possibility of stating a similar result for the space ℓ_1. Since weak compactness is equivalent to compactness for subsets of ℓ_1, it is clear that every weakly compact convex subset of ℓ_1 satisfies the FPP for nonexpansive mappings. However, it is well known that there are many convex noncompact subsets of ℓ_1 (for instance, weak* compact sets) which satisfy the FPP for nonexpansive mappings (more detailed information about subsets of ℓ_1 satisfying the FPP for nonexpansive mappings can be found in [8]). In fact, K. Goebel and T. Kuczumow [15] constructed a nested sequence of convex closed subsets of the space ℓ_1 which alternatively satisfy or fail the FPP for nonexpansive mappings. This sharp example seemed to point to that compact sets are the only convex closed subsets of ℓ_1 that satisfy the hereditary FPP for nonexpansive mappings (i.e. every closed convex subset satisfies the FPP for nonexpansive mappings). This assertion was, in fact, proved by P. Dowling et al. [10] in case that the set is norm-bounded.

In this paper we will give a characterization of norm compactness in ℓ_1 where, firstly the boundedness condition is not longer required in the hypothesis and secondly, we can also drop the hereditary assumption because this characterization can be achieved by means of the existence of fixed points for certain families of self-mappings defined over the set C itself, in contrast to the results in [10]. In particular, in case of nonexpansive mappings, we prove that compactness for a closed convex subset of ℓ_1 is equivalent to satisfy the FPP for Lipschitzian mappings which are nonexpansive on their ranges.

Furthermore, we could wonder if Theorem 1.1 is also true for uniformly Lipschitzian mappings, i.e. mappings such that all iterates are L-Lipschitzian. Looking at the literature on fixed points for uniformly Lipschitzian mappings (see, for instance, [5,6,9,16,22,23]), it is very clear that this is not, in general, the case, because for small values of $L > 1$ it is possible to obtain the existence of a fixed point in convex bounded closed subsets of several classes of Banach spaces. For instance, for $X = \ell_2$, uniformly L-Lipschitzian mappings defined from a convex bounded closed subset C into C have a fixed point if $L < \sqrt{2}$ [22]. In spite of these existence results for noncompact sets and small values of L, we can still obtain a characterization of