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Positive solutions of logistic equations with dependence on gradient and nonhomogeneous Kirchhoff term

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ABSTRACT

Consider the equation $-M(x,\|u\|)\Delta_p u = \lambda f(x,u,\nabla u) - g(x,u,\nabla u)$ in Ω, $u = 0$ on $\partial\Omega$. The main aim of this paper is to prove existence results for both non-degenerate and degenerate cases of the function M. Our approach relies on the fixed point index and the cone theoretic argument.

1. Introduction

In the paper we consider the following elliptic equation containing the Kirchhoff term

$$\begin{cases} -M\left(x, \|u\|\right)\Delta_p u = \lambda f(x,u,\nabla u) - g(x,u,\nabla u) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain with a smooth boundary in $\mathbb{R}^N$, $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian, $1 < p < N$, $\|\cdot\|$ denotes the norm in Sobolev space $W_0^{1,p}(\Omega)$, λ is a real parameter and $M : \Omega \times \mathbb{R}^+ \to \mathbb{R}^+$, $f, g : \Omega \times \mathbb{R}^+ \times \mathbb{R}^N \to \mathbb{R}^+$ are suitable functions.

When $M(x,t) \equiv 1$ and $f(x,u,v) = a(x)u^\alpha$, $g(x,u,v) = b(x)u^\beta$ with $1 < \alpha < \beta$, the equation (1.1) is called a p-logistic equation. Note that the theory of p-logistic equation has many important applications in

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reaction–diffusion processes and in biological models. This also has received a great deal of attention from mathematicians. See for example [8–11,15,28,29] and references therein.

Recently, in [5,16], using the variational method, the authors investigated the logistic equations in the more general form $-\Delta_p u = \lambda f(x,u) - g(x,u)$. Moreover, the logistic equations with the nonlinearities depending on the gradient were studied in [17,18,26]. Note that in these works, the topological degree arguments and bifurcation theory were used as a main technical element.

Equations (1.1) is the stationary problem associated to the time-dependent problem

$$\begin{cases} u_{tt} - M(x, \|u\|)\Delta u = f & \text{in } \Omega \times (0,T), \\ u = 0 & \text{on } \partial\Omega \times (0,T), \\ u(x,0) = u_0(x), u_t(x,0) = u_1(x) & \text{in } \Omega, \end{cases}$$

which models small vertical vibration of an elastic string when the density of the material is not constant [23]. Elliptic equations with Kichhoff term $M(x, \|u\|)$ have been investigated in past decade intensively. See e.g. [2,6,7,12,19–21,24,27,31–33] and references therein. However, most of the articles on this subject are concerned with the case when the function f in (1.1) does not depend on ∇u and $g = 0$ and with the homogeneous Kirchhoff term M, i.e. M does not depend on x, $M(x,t) = a + bt^2$, where a, b are positive constants. Under these assumptions, the equations have a variational structure. Recently, the case of dependence on the gradient and nonhomogeneous Kirchhoff term have been considered. The authors in [1] studied the equation (1.1) with $p = 2$, $\lambda = 1$, $M = M(x, \|u\|)$ and $g = 0$ by sub-supersolution method and then applied to consider the case of nonlinearities $Au^\alpha(B-u) + |\nabla u|^\gamma, \lambda u^\alpha + u^\beta + \mu|\nabla u|^\gamma$. The equation in the following form was investigated in [13]:

$$-(a(x) + b(x)\|u\|^2)\Delta u = \lambda u^q \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \partial\Omega, \tag{1.2}$$

where $q \in (0,1]$, $a(x) \geq a_0 > 0$, $b(x) \geq 0$ and $int\{x \in \Omega : b(x) = 0\}$ is a regular sub-domain of Ω. It was proved that in the non-homogeneous case of the term $b(x)$, equation (1.2) has a positive solution only for $\lambda \in (\lambda_0, \lambda_1)$, while in the homogeneous case, equation (1.2) has a positive solution for $\lambda > \lambda_0$.

The purpose of our paper is to extend the existence results of the work [13] to the equation (1.1). Using the fixed point index theory in conjunction with the cone theoretic argument we are able to prove the following results.

(a) In the "non-degenerate" case of the function M, the equation (1.1) has a positive solution if either $\lambda > 0$ and f is $(p-1)$-sublinear corresponding to the second variable or $\lambda > \lambda_1$ and f is $(p-1)$-linear corresponding to the second variable.
(b) In the case of "degenerate" of the function M, the equation (1.1) has a positive solution for $\lambda \in (\lambda_1, \lambda_2)$ and has no positive solution for $\lambda > \lambda_2 + \sigma$ as f is $(p-1)$-linear corresponding to the second variable.

Here, $\lambda_1, \lambda_2, \sigma$ are constants in Theorem 3.5.

Note that in the particular case when $M(x,t) = a(x) + b(x)t^2$, $f(x,u,v) = u^q$, $g = 0$ and $p = 2$, we recover the results in [13] on the existence results. In the general case, our results are new in the literature. Moreover, in comparison with the methods in [13], our approach is different. In this paper, we combine the fixed point index theory with the cone theoretic argument which are key technical ingredients to obtain the main results.

The paper is organized as follows. In section 2 we reduce equation (1.1) to a fixed point problem. Section 3 is devoted to the main results of the paper.