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On the relationship between the lower order of coefficients and the growth of solutions of differential equations ☆

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ABSTRACT

Some criteria for entire coefficients $A(z)$ and $B(z)$ are given in terms of the lower order forcing the solutions of $f''+A(z)f'+B(z)f=0$ to grow fast. In the literature similar criteria have been published in terms of the usual order. The case when the coefficient $A(z)$ has an asymptotic growth $T(r,A)\sim\alpha\log M(r,A)$, $\alpha\in(0,1)$, outside of an exceptional set is also discussed. Previously, Laine–Wu (2000) and Kim–Kwon (2001) have made use of this asymptotic growth in the case $\alpha=1$.

1. Introduction and main results

For a function f meromorphic in the complex plane $\mathbb{C}$, the order of growth is given by

$$\rho(f)=\limsup_{r\to\infty}\frac{\log^+ T(r,f)}{\log r}.$$

If f is entire, then the Nevanlinna characteristic $T(r,f)$ can be replaced with $\log M(r,f)$, where $M(r,f)=\max_{|z|=r}|f(z)|$. The lower order $\mu(f)$ of f is defined similarly but for "lim inf" instead of "lim sup". These orders give upper and lower restrictions for the growth of $T(r,f)$ or of $\log M(r,f)$.

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It is clear that $\mu(f) \leq \rho(f)$ holds in general, and that the strict inequality is possible, see [4, p. 314], [5, p. 439] or [11, Theorem 4]. In fact, for any fixed μ and ρ satisfying $0 \leq \mu \leq \rho \leq \infty$ there exists an entire function with order ρ and lower order μ [6, p. 238]. If $\mu(f) < \rho(f)$ holds for an entire function f, set $\mu(f) < a < b < \rho(f)$, and define

$$\begin{aligned} E &= \{r \geq 0 : T(r,f) < r^a\}, \\ F &= \{r \geq 0 : T(r,f) > r^b\}. \end{aligned} \tag{1}$$

Then Corollary 3.7 below states that the sets E and F are of lower density zero and of upper density one. Densities of sets will be recalled in Section 3.

The concept of lower order is often associated with Pólya peaks and the spread relation [25, pp. 217–232], the deficiency problem [25, pp. 251–255], as well as Julia directions [27, pp. 282–304]. The concept of (usual) order has a wide range of applications including complex differential equations, see [14,15]. Some attempts to associate the lower order with complex differential equations exist, see [17,23,24].

There are many results in the literature for the growth of solutions under certain restrictions for the coefficient functions in terms of the usual order. The main theme in this paper is to show that very similar or even the exact same conclusions on the solutions can be made if the usual order is replaced with the lower order. Our starting point is a result due to Gundersen.

Theorem A. [8] *Let $A(z)$ and $B(z)$ be two entire functions with $\rho(A) < \rho(B)$. Then every nontrivial solution of the equation*

$$f'' + A(z)f' + B(z)f = 0 \tag{2}$$

is of infinite order.

Our response to Theorem A is that the usual orders can simply be replaced with the corresponding lower orders.

Theorem 1.1. *Let $A(z)$ and $B(z)$ be two entire functions with $\mu(A) < \mu(B)$. Then every nontrivial solution of (2) is of infinite order.*

For (2) to possess a nontrivial solution of finite order, it is therefore necessary that $\mu(B) \leq \mu(A)$. Here either an equality or a strict inequality can hold, see Example 2.1 below.

Theorem A leaves a question what might happen if $\rho(B) \leq \rho(A)$. Since $f(z) = -z$ solves $f'' - ze^z f' + e^z f = 0$, we see that polynomial solutions are possible in the case $\rho(B) = \rho(A)$. It is known that if $A(z)$ and $B(z)$ are entire with $\rho(B) < \rho(A)$, then every nontrivial solution f of (2) satisfies $\rho(f) \geq \rho(A)$ [12, Theorem 2]. As a consequence of this reasoning, we show that the usual order can be replaced with the lower order throughout here. In addition, Part (ii) of Example 2.1 shows that the equality $\mu(f) = \mu(A)$ is possible under the assumption $\mu(B) < \mu(A)$.

Corollary 1.2. *Let $A(z)$ and $B(z)$ be two entire functions with $\mu(B) < \mu(A)$. Then every nontrivial solution f of (2) satisfies $\mu(f) \geq \mu(A)$.*

The previous example $f'' - ze^z f' + e^z f = 0$ with a polynomial solution $f(z) = -z$ also shows that the conclusion in Corollary 1.2 is false if $\mu(A) = \mu(B)$. Next we quote another result due to Gundersen.

Theorem B. [8] *Let $A(z)$ and $B(z)$ be two entire functions such that for real constants $\alpha, \beta, \theta_1, \theta_2$, where $\alpha > 0, \beta > 0$ and $\theta_1 < \theta_2$, we have*