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Proof of Sun's conjectures on integer-valued polynomials

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ABSTRACT

Recently, Z.-W. Sun introduced two kinds of polynomials related to the Delannoy numbers, and proved some supercongruences on sums involving those polynomials. We deduce new summation formulas for squares of those polynomials and use them to prove that certain rational sums involving even powers of those polynomials are integers whenever they are evaluated at integers. This confirms two conjectures of Z.-W. Sun. We also conjecture that many of these results have neat q-analogues.

1. Introduction

It is well known that, for any $m, n \geqslant 0$, the number

$$\sum_{k=0}^{n}\binom{n}{k}\binom{m}{k}2^k=\sum_{k=0}^{n}\binom{n}{k}\binom{n+m-k}{n},$$

called a *Delannoy number*, counts lattice paths from $(0,0)$ to (m,n) in which only east $(1,0)$, north $(0,1)$, and northeast $(1,1)$ steps are allowed. Recently, Z.-W. Sun [15] introduced the following polynomials

$$d_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}2^k,$$

$$s_n(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{x}{k}\binom{x+k}{k},$$

and established some interesting supercongruences involving $d_n(x)$ or $s_n(x)$, such as

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$$\sum_{k=0}^{p-1}(2k+1)d_k(x)^2 \equiv \begin{cases} -x \pmod{p^2}, & \text{if } x \equiv 0 \pmod{p}, \\ x+1 \pmod{p^2}, & \text{if } x \equiv -1 \pmod{p}, \\ 0 \pmod{p^2}, & \text{otherwise,} \end{cases} \tag{1.1}$$

$$\sum_{k=0}^{p-1}(2k+1)s_k(x)^2 \equiv 0 \pmod{p^2}, \tag{1.2}$$

where p is an odd prime and x is a p-adic integer.

Recall that a polynomial $P(x)$ in x with real coefficients is called *integer-valued*, if $P(x) \in \mathbb{Z}$ for all $x \in \mathbb{Z}$. In this paper, we shall prove the following generalizations of (1.1) and (1.2), which were originally conjectured by Z.-W. Sun (see [15, Conjectures 6.1 and 6.12]).

Theorem 1.1. *Let m and n be positive integers. Then all of*

$$\frac{x(x+1)}{2n^2}\sum_{k=0}^{n-1}(2k+1)d_k(x)^2, \quad \frac{1}{n}\sum_{k=0}^{n-1}(2k+1)d_k(x)^{2m}, \quad \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)d_k(x)^{2m},$$
$$\frac{1}{2n^2}\sum_{k=0}^{n-1}(2k+1)s_k(x)^2, \quad \frac{1}{n}\sum_{k=0}^{n-1}(2k+1)s_k(x)^{2m}, \quad \frac{1}{n}\sum_{k=0}^{n-1}(-1)^k(2k+1)s_k(x)^{2m}$$

are integer-valued.

We shall also prove the following result, which will play an important role in our proof of Theorem 1.1.

Theorem 1.2. *Let m and n be positive integers and let j, k be non-negative integers. Then*

$$\frac{(n-k)(k+1)}{n}\binom{n+k}{2k}\binom{m+1}{k+1}\binom{m+k}{k+1}$$

and

$$\frac{1}{k+1}\binom{n-1}{k}\binom{n+k}{k}\binom{2k}{j+k}\binom{m+k}{2k}\binom{m}{j}\binom{m+j}{j}$$

are integers.

The paper is organized as follows. In the next section, we shall give a q-analogue of Theorem 1.2. In Section 3, we mainly give a single-sum expression for $d_n(x)^2$, a new expression for $s_n(x)^2$, and recall a recent divisibility result of Chen and Guo [3] concerning multi-variable Schmidt polynomials. The proof of Theorem 1.1 will be given in Section 4. We propose some related open problems in the last section.

2. A q-analogue of Theorem 1.2

Recall that the *q-binomial coefficients* are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{cases} \displaystyle\prod_{i=1}^{k}\frac{1-q^{n-k+i}}{1-q^i}, & \text{if } 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

The following is our announced strengthening of Theorem 1.2.