J. Math. Anal. Appl. 444 (2016) 326–339

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Quantitative uniqueness of some higher order elliptic equations

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ARTICLE INFO

Article history:
Received 14 January 2016
Available online 22 June 2016
Submitted by X. Zhang

Keywords:
Quantitative uniqueness
Higher order elliptic operators
Carleman estimates

ABSTRACT

We study the quantitative unique continuation property of some higher order elliptic operators. We prove a lower bound for nontrivial solutions of the equation $(-\Delta)^m u + V(x)u = 0$, $m \in \mathbb{N}$, $V(x)$ is bounded. The bound shows that nontrivial solutions can not decay faster than $e^{-|x|^{4/3}\ln|x|}$ at infinity. Moreover, we obtain an improved lower bound of nontrivial solutions for a special forth order elliptic operators in dimension 2, the bound is shown to be essentially sharp by constructing a Meshkov-type example.

1. Introduction

In this paper, we are interested in the following quantitative unique continuation problem at infinity for some higher order elliptic operators with constant coefficients. Suppose that u satisfies

$$P(D)u + Vu = 0 \quad \text{in} \quad \mathbb{R}^n, \tag{1.1}$$

and

$$|V(x)| \leq C, \quad |u(x)| \leq C, \quad u(0) = 1, \quad x \in \mathbb{R}^n. \tag{1.2}$$

For large R, one can define

$$M(R) = \inf_{|x_0|=R} \sup_{B(x_0,1)} |u(x)|$$

to measure the precise least decay information at infinity of the solution. Then a natural question is how small can $M(R)$ be? We first briefly recall the second order case. In 1960, Landis [11] conjectured that

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http://dx.doi.org/10.1016/j.jmaa.2016.06.043
0022-247X/

if $u(x)$ is a solution of (1.1) and (1.2) with $P = \Delta = \partial_1^2 + \partial_2^2 + \cdots + \partial_n^2$, $u(x) \le C\exp\{-C|x|^{1+}\}$ for some constant, then u is identically zero. Meshkov [15] disproved this conjecture by constructing non-trivial bounded, complex-valued functions u, V satisfying (1.1), (1.2) and $u(x) \lesssim e^{-C|x|^{\frac{4}{3}}}$. In 2005, Bourgain and Kenig [2] derived a quantitative version of Meshkov's result in their resolution of Anderson localization for the Bernoulli model. More precisely, they showed that the non-trivial solution of (1.1) and (1.2) with $P = \Delta$ satisfies the bound $M(R) \gtrsim \exp\{-CR^{\frac{4}{3}}\log R\}$. This lower bound is sharp in view of Meshkov's example [15]. We mention that a Meshkov-type (nontrivial solution) construction for elliptic equations with gradient terms was given by Duyckaerts, Zhang and Zuazua in [6]. Moreover, the quantitative unique continuation for second order elliptic equations with gradient term were obtained in Davey [5], Lin and Wang [14].

Now we turn to the higher order case. Weak and strong unique continuation properties for higher order elliptic equations have been studied by many authors, see e.g. [3,4,10,13] and references therein. Note that Alinhac's counterexample (see e.g. [1]) indicates that in general the strong unique continuation property may fail. Lerner [12] showed in dimension 2 that if coefficients are in the Gevrey class of small index, the strong unique continuation property still holds. The result was extended to higher dimensions by Colombini, Grammatico, Tataru [4]. It seems that there is very limited number of results available regarding quantitative results mentioned above for higher order elliptic equations. To our best of knowledge, the only previous result was obtained recently by Zhu [19], who proved a vanishing order estimate for solutions to equations whose leading operator is a power of the Laplacian. The main ingredient in [19] is to exploit the monotonicity property of a variant of frequency functions, where its application to strong unique continuation problems was first observed by Garofalo and Lin [7]. As a corollary, it was shown that for $P = (-\Delta)^m$, $m \in \mathbb{N}$, and if u is a solution to (1.1) with $n \ge 4m$, then for each $R > 0$

$$M(R) \gtrsim \exp\left\{-CR^{2m}\log R\right\}.$$

We shall remove the condition $n \ge 4m$ and obtain an improved lower bound. Instead of using frequency functions and Sobolev estimates, a simple iteration of the Carleman estimates used in [2] will allow us to follow Bourgain and Kenig's approach. Our first result is

Theorem 1.1. *Let $P = (-\Delta)^m$, $m \in \mathbb{N}^+$, and assume that u satisfies (1.1) and (1.2). Then for each $R > 0$*

$$M(R) \gtrsim \exp\left\{-CR^{\frac{4}{3}}\log R\right\}.$$

Currently, we don't know yet whether the exponent $\frac{4}{3}$ here is optimal (up to logarithmic loss) for $(-\Delta)^m$, $m > 1$.[1] Nevertheless, in dimension 2, we are able to obtain better lower bounds for $P = \Delta_{\mathbb{R}^2}(\partial_{x_1}^2 + (1+\frac{\epsilon}{2})\partial_{x_2}^2)$ with $\epsilon > 0$.

Theorem 1.2. *For any given $\epsilon > 0$, let $P_\epsilon = P_1 P_{2,\epsilon}$ on $\mathbb{R}^2$, where $P_1 = \Delta_{\mathbb{R}^2}$ and $P_{2,\epsilon} = \partial_{x_1}^2 + (1 + \frac{\epsilon}{2})\partial_{x_2}^2$. Assume that u satisfies (1.1) and (1.2). Then for each $R > 0$*

$$M(R) \gtrsim_\epsilon \exp\left\{-CR^{\frac{8}{7}+\epsilon}\right\}.$$

Furthermore, we shall construct a Meshkov-type example to show that the result obtained in Theorem 1.2 is essentially sharp (up to ϵ-power loss).

Theorem 1.3. *Assume that P satisfies the assumptions of Theorem 1.2. There exist nontrivial bounded functions u and V satisfying (1.1) and (1.2) such that*

[1] In the case $m = 2$, it was claimed in [18] that the exponent $\frac{4}{3}$ is sharp.