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Blow-up results for evolution problems with inhomogeneous nonlocal diffusion ☆

Yujuan Chen [a], Yueping Zhu [b,*]

[a] School of Science, Nantong University, Nantong, 226007, PR China
[b] Mathematics department, Nantong Normal College, Nantong, 226001, PR China

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ABSTRACT

We provide a sufficient condition on the existence and nonexistence of global positive solutions to the Cauchy problem for an inhomogeneous nonlocal diffusion $u_t = J * u - u + u^p + f(x)$, where J is a nonnegative function, $p > 0$, and $f \geq 0, \not\equiv 0$. Meanwhile, the case of an inhomogeneous nonlocal diffusion system is considered. It turns out that the inhomogeneous terms substantially contribute to the blow-up exponent, which coincides with the classical one when the diffusion is given by the Laplacian.

1. Introduction and main results

In this paper, we are concerned with the following evolution problem

$$\begin{cases} u_t = J * u - u + u^p + f(x), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where $*$ stands for the usual convolution, namely, $J * u(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$, $p > 0$, $u_0, f \geq 0, \not\equiv 0$, and $f, u_0 \in L^1(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$. Throughout the paper we will always assume $J \in C(\mathbb{R}^N)$ is compactly supported, radially symmetric, nonnegative and $\int_{\mathbb{R}^N} J(z)dz = 1$.

As we know, nonlocal evolution equations like (1.1) have been recently widely used in the modeling of diffusion processes, see [2–4,8,12]. As stated in [8], if $u(x,t)$ is thought of as the density of a single population at the point x at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from location y to location x, then $(J * u)(x,t) = \int_{\mathbb{R}^N} J(x-y)u(y,t)dy$ is the rate at which individuals are arriving to position x from all other places and $-u(x,t) = -\int_{\mathbb{R}^N} J(y-x)u(x,t)dy$ is the rate at which they are leaving

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* Corresponding author.
E-mail address: zhuyueping@ntu.edu.cn (Y. Zhu).

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location x to travel to all other sites. Eq. (1.1) is then called nonlocal diffusion equation since the diffusion of the density u at a point x and time t does not only depend on $u(x,t)$, but on all the values of u in a neighborhood of x through the convolution term $J * u$.

Let us provide some motivations for investigating problems of the form (1.1). In [9], Fujita proved the following results for the problem

$$\begin{cases} u_t = \Delta u + u^p, & x \in \mathbb{R}^N,\ t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N. \end{cases} \tag{1.2}$$

(a) When $1 < p < 1 + \frac{2}{N}$ and $u_0 > 0$, the problem (1.2) possesses no global positive solution.
(b) When $p > 1 + \frac{2}{N}$ and u_0 is smaller than a small Gaussian, then (1.2) has global positive solutions.

When $p = 1 + \frac{2}{N}$, it was shown later in [1,13] that all nontrivial nonnegative solutions to (1.2) blow up in finite time, so $p_c := 1 + \frac{2}{N}$ is the critical exponent of (1.2), that is what is usually known as the Fujita exponent. These elegant works revealed a new phenomenon of nonlinear PDEs and stimulated the study of similar features for various nonlinear evolution equations (see, e.g. the survey papers [14] and the references therein, and also the recent papers [5,10,11,20]).

As for the nonlocal diffusion, recently, J. García-melián et al. [11] proved that the Fujita exponent for the problem

$$\begin{cases} u_t = J * u(x,t) - u(x,t) + u^p(x,t), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.3}$$

is $p_c = 1 + \frac{2}{N}$, which coincides with the one for the problem (1.2).

While for coupled system, Escobedo and Herrero [6], Qi and Levine [17] proved the boundedness and blow-up for the Cauchy problem

$$u_t = \Delta u^m + v^p, \quad v_t = \Delta v^n + u^q, \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.4}$$

with $0 < m, n \le 1$, $p, q \ge 1$, $pq > 1$, and got the critical Fujita curve $(pq)_c = mn + \frac{2}{N} \max\{p + n, q + m\}$, namely, every solution blows up in finite time if $1 < pq \le (pq)_c$, and there exist both global and non-global solutions if $pq > (pq)_c$. For a nonlinear coupled nonlocal diffusion system

$$\begin{cases} u_t = J * u - u + v^p, & x \in \mathbb{R}^N, t > 0, \\ v_t = J * v - v + u^q, & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases}$$

Yang [18] established that the critical Fujita curve is the same as that of (1.4) with $m = n = 1$. Very recently, Mohamed [16] provided a sufficient condition on the nonexistence of global positive solutions to the nonlocal fractional diffusion problem.

However, for the inhomogeneous Cauchy problem,

$$u_t = \Delta u + u^p + f(x), \quad x \in \mathbb{R}^N,\ t > 0, \tag{1.5}$$

Zhang [20] obtained that the critical exponent of (1.5) is $p_c = 1 + \frac{2}{(N-2)_+}$, where $(N-2)_+$ denotes the positive part of $N - 2$. Correspondingly, for the inhomogeneous coupled system

$$\begin{cases} u_t = \Delta u^m + v^p + f_1(x), & x \in \mathbb{R}^N, t > 0, \\ v_t = \Delta v^n + u^q + f_2(x), & x \in \mathbb{R}^N, t > 0, \\ u(x,0) = u_0(x), v(x,0) = v_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$