J. Math. Anal. Appl. 444 (2016) 475–502

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Solution of multilayer diffusion problems via the Laplace transform

Marianito R. Rodrigo, Annette L. Worthy *

School of Mathematics and Applied Statistics, University of Wollongong, Wollongong, New South Wales, Australia

ARTICLE INFO

Article history:
Received 5 January 2016
Available online 27 June 2016
Submitted by Y. Du

Keywords:
Multilayer
Diffusion
Laplace transform
Renewal equation

ABSTRACT

We consider a one-dimensional multilayer diffusion problem subject to nonhomogeneous boundary conditions. Unlike previous results that used a separation of variables technique to solve such problems with homogeneous boundary conditions, here we use a Laplace transform approach. We reformulate the multilayer diffusion problem as a sequence of one-layer diffusion problems with arbitrary time-dependent functions, solve a general one-layer diffusion problem using the Laplace transform, and then use the interface conditions to determine a system of renewal-type equations for the time-dependent functions. Finally, these renewal equations are solved explicitly using the Laplace transform.

1. Introduction

Multilayer diffusion problems arise in many applications of heat and mass transfer. Some industrial applications are washing of wool [4] and moisture diffusion in woven fabric composites [16]. Applications to medicine include skin permeability [15], drug-eluting stents [17,12], and drug delivery [2,19]. Environmental applications include contamination in a porous medium [9,10] and chamber-based gas fluxes measurements [11]. More references to applications can be found in [7].

There are two main approaches to finding exact solutions of multilayer diffusion problems: separation of variables and integral transforms. Separation of variables is used when the outer boundary conditions (BCs) are homogeneous (see [5,7,13,14,17,18] and the references therein). The requirement of orthogonality is extended in order to incorporate the inner BCs (or interface conditions). On the other hand, the method of Laplace transforms is applicable even for nonhomogeneous BCs, but is not as common as the separation of variables technique due to the difficulty of the inversion step. This difficulty is compounded when there are more than two layers (see [12,19] for the case of two layers). Frequently, the Laplace inversion is done numerically [1].

* Corresponding author.
E-mail addresses: marianito_rodrigo@uow.edu.au (M.R. Rodrigo), annette_worthy@uow.edu.au (A.L. Worthy).

http://dx.doi.org/10.1016/j.jmaa.2016.06.042
0022-247X/

In a recent article, Carr and Turner [3] found a semi-analytical solution for a multilayer problem via the Laplace transform. The authors considered a multilayer diffusion problem with both perfect and imperfect contact at the interfaces between adjacent layers. The outer BCs were nonhomogeneous but constant. They used a hybrid approach based on the Laplace transform and orthogonal eigenfunction expansions. The solution they found was semi-analytical in that it had terms that involved inverse Laplace transforms that had to be evaluated numerically.

In this article we solve a multilayer diffusion problem using the Laplace transform. We consider nonhomogeneous outer BCs with arbitrary time-varying functions. Instead of applying the Laplace transform to each diffusion equation and attempting to find a general Laplace inversion formula, our solution procedure is to first consider a one-layer diffusion problem with BCs that involve arbitrary time-dependent functions and solve this problem completely with the Laplace transform. Then we express the multilayer diffusion problem as a sequence of one-layer diffusion problems with appropriate BCs that have arbitrary time-dependent functions. We use the analytical solution of the one-layer problem to find the solution of the multilayer problem by determining the time-dependent functions through the interface conditions. This leads to a system of renewal-type equations whose solution can again be obtained via the Laplace transform. Although we assume perfect contact conditions at the interfaces, our approach can be similarly adapted for imperfect contact conditions by an appropriate modification of the renewal-type equations.

This paper is organized as follows. In Section 2 we give the mathematical formulation of a multilayer diffusion problem and then reformulate it as a sequence of one-layer diffusion problems with BCs that include arbitrary time-dependent functions. This reformulation motivates the consideration of a general one-layer problem with time-varying BCs, which we solve in Section 3. The solution is achieved with the help of four lemmas whose proofs are relegated to the Appendix. Two examples are presented to illustrate the result. In Section 4 we consider the particular case of the two-layer problem to highlight the essential ideas and give a particular illustrative example. In Section 5 we return to the multilayer diffusion problem and use the results of Sections 3 and 4 to find the solution. Finally, we give a discussion on the determination of critical times and brief concluding remarks in Section 6.

2. Mathematical formulation of a multilayer diffusion problem

A multilayer diffusion problem is set out as follows. Let $a, b \in \mathbb{R}$, where $a < b$, and suppose that we have a partition

$$a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b$$

of $[a, b]$. On each subinterval $[x_{j-1}, x_j]$, where $j = 1, \ldots, n$, we consider the partial differential equation (PDE)

$$\frac{\partial u_j}{\partial t} = d_j \frac{\partial^2 u_j}{\partial x^2}, \quad (x, t) \in (x_{j-1}, x_j) \times \mathbb{R}_+ \tag{2.1}$$

for each component function u_j for $j = 1, \ldots, n$. Here, $\mathbb{R}_+ = (0, \infty)$ and d_j for all $j = 1, \ldots, n$ are positive diffusion coefficients. Furthermore, the initial conditions (ICs) are

$$u_j(x, 0) = f_j(x), \quad x \in [x_{j-1}, x_j] \tag{2.2}$$

for all $j = 1, \ldots, n$. The outer BCs are given by

$$\alpha u_1(a, t) + \beta \frac{\partial u_1}{\partial x}(a, t) = g(t), \quad \gamma u_n(b, t) + \delta \frac{\partial u_n}{\partial x}(b, t) = h(t), \quad t \in \mathbb{R}_+, \tag{2.3}$$