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Global bifurcations close to symmetry ☆

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Heteroclinic cycles involving two saddle-foci, where the saddle-foci share both invariant manifolds, occur persistently in some symmetric differential equations on the 3-dimensional sphere. We analyse the dynamics around this type of cycle in the case when trajectories near the two equilibria turn in the same direction around a 1-dimensional connection — the saddle-foci have the same chirality. When part of the symmetry is broken, the 2-dimensional invariant manifolds intersect transversely creating a heteroclinic network of Bykov cycles. We show that the proximity of symmetry creates heteroclinic tangencies that coexist with hyperbolic dynamics. There are n-pulse heteroclinic tangencies — trajectories that follow the original cycle n times around before they arrive at the other node. Each n-pulse heteroclinic tangency is accumulated by a sequence of $(n+1)$-pulse ones. This coexists with the suspension of horseshoes defined on an infinite set of disjoint strips, where the first return map is hyperbolic. We also show how, as the system approaches full symmetry, the suspended horseshoes are destroyed, creating regions with infinitely many attracting periodic solutions.

1. Introduction

A Bykov cycle is a heteroclinic cycle between two hyperbolic saddle-foci of different Morse index, where one of the connections is transverse and the other is structurally unstable — see Fig. 1. There are two types of Bykov cycle, depending on the way the flow turns around the two saddle-foci, that determine the *chirality* of the cycle. Here we study the non-wandering dynamics in the neighbourhood of a Bykov cycle where the two nodes have the same chirality. This is also studied in [34], and the case of different chirality is discussed in [36]. A simplified version of the arguments presented here appears in [35].

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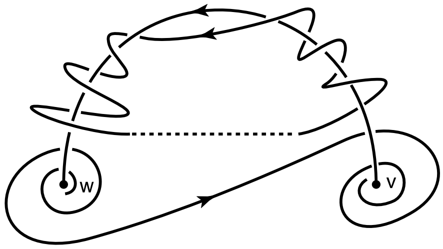

Fig. 1. A Bykov cycle with nodes of the same chirality. There are two possibilities for the geometry of the flow around a Bykov cycle depending on the direction trajectories turn around the connection $[\mathbf{v} \to \mathbf{w}]$. We assume here that the nodes have the same chirality: trajectories turn in the same direction around the connection. When the endpoints of a nearby trajectory are joined, the closed curve is always linked to the cycle.

1.1. The object of study

Our starting point is a fully $(\mathbf{Z}_2 \times \mathbf{Z}_2)$-symmetric differential equation $\dot{x} = f_0(x)$ in the three-dimensional sphere $\mathbf{S}^3$ with two saddle-foci that share all the invariant manifolds, of dimensions one and two, both contained in flow-invariant submanifolds that come from the symmetry. This forms an attracting heteroclinic network Σ^0 with a non-empty basin of attraction V^0. We study the global transition of the dynamics from this fully symmetric system $\dot{x} = f_0(x)$ to a perturbed system $\dot{x} = f_\lambda(x)$, for a smooth one-parameter family that breaks part of the symmetry of the system. For small perturbations the set V^0 is still positively invariant.

When $\lambda \neq 0$, the one-dimensional connection persists, due to the remaining symmetry, and the two dimensional invariant manifolds intersect transversely, because of the symmetry breaking. This gives rise to a network Σ^λ, that consists of a union of Bykov cycles, contained in V^0. For partial symmetry-breaking perturbations of f_0, we are interested in the dynamics in the maximal invariant set contained in V^0. It contains, but does not coincide with, the suspension of horseshoes accumulating on Σ^λ described in [3,31,34,46]. Here, we show that close to the fully symmetric case it contains infinitely many heteroclinic tangencies. Under an additional assumption, we show that V_0 contains attracting limit cycles with long periods, coexisting with sets with positive entropy.

1.2. History

Homoclinic and heteroclinic bifurcations constitute the core of our understanding of complicated recurrent behaviour in dynamical systems. This starts with Poincaré on the late 19th century, with major subsequent contributions by the schools of Andronov, Shilnikov, Smale and Palis. These results rely on a combination of analytical and geometrical tools used to understand the qualitative behaviour of the dynamics.

Heteroclinic cycles and networks are flow-invariant sets that can occur robustly in dynamical systems with symmetry, and are frequently associated with intermittent behaviour. The rigorous analysis of the dynamics associated to the structure of the nonwandering sets close to heteroclinic networks is still a challenge. We refer to [28] for an overview of heteroclinic bifurcations and for details on the dynamics near different kinds of heteroclinic cycles and networks.

Bykov cycles have been found analytically in the Lorenz model in [1,44] and the nearby dynamics was studied by Bykov in [9–11]. The point in parameter space where this cycle occurs is called a *T-point* in [18]. Recently, there has been a renewal of interest in this type of heteroclinic bifurcation in the reversible [15,16,37], equivariant [3,34,48] and conservative cases [7]. See also [31].

The transverse intersection of the two-dimensional invariant manifolds of the two equilibria implies that the set of trajectories that remain for all time in a small neighbourhood of the Bykov cycle contains a locally-maximal hyperbolic set admitting a complete description in terms of symbolic dynamics, reminiscent of the results of Shilnikov [52]. An obstacle to the global symbolic description of these trajectories is the