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Reverse order laws for the Drazin inverses

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A B S T R A C T

This paper is to present some equivalent conditions concerning the reverse order law $(PQ)^D = Q^D P^D$ for the Drazin invertible operators P and Q with partial commutative properties $[P, PQ] = 0$ or $[Q, PQ] = 0$ or $[P, PQQ^D] = 0$ or $[Q, PP^D Q] = 0$.

1. Introduction

Let $\mathcal{B}(\mathcal{H})$ denote the set of all bounded linear operators on a complex Hilbert space $\mathcal{H}$. For $P \in \mathcal{B}(\mathcal{H})$, denote by P^*, $\mathcal{N}(P)$ and $\mathcal{R}(P)$ the adjoint, the null space and the range of P, respectively. If P, $Q \in \mathcal{B}(\mathcal{H})$, then

$$[P, Q] =: PQ - QP$$

denotes the commutator of P and Q. Commutators arise naturally in many aspects of operator theory, and they play an important role in this theory. It is well known that the set of commutators is dense in the set of all operators [2, page. 124]. Commutation relations between operators play an important role in the representations of the Drazin inverses. Some properties of the Drazin inverses according to such relations have been extensively studied in the mathematical literature (see, e.g., [7] and [8]). It is well known that, if $PQ = QP = 0$, Drazin in his celebrated paper [8] had proved that

$$(P - Q)^D = P^D - Q^D.$$

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We adopt the following definitions and notations. An operator $T \in \mathcal{B}(\mathcal{H})$ is said to be idempotent if $T^2 = T$. For $T \in \mathcal{B}(\mathcal{H})$, if there exists an operator $T^D \in \mathcal{B}(\mathcal{H})$ satisfying the following three operator equations [8]

$$TT^D = T^D T, \qquad T^D T T^D = T^D, \qquad T^{k+1} T^D = T^k, \tag{1}$$

where $k = ind(T)$, the index of T, is the smallest nonnegative integer for which $\mathcal{R}(T^{k+1}) = \mathcal{R}(T^k)$ and $\mathcal{N}(T^{k+1}) = \mathcal{N}(T^k)$, then T^D is called a Drazin inverse of T. The conditions (1) are equivalent to

$$TT^D = T^D T, \qquad T^D T T^D = T^D, \qquad T - T^2 T^D \text{ is nilpotent.}$$

If T is Drazin invertible (for short DI), then the spectral idempotent P^π of T corresponding to $\{0\}$ is given by $P^\pi = I - TT^D$. The operator matrix form of T with respect to the space decomposition $\mathcal{H} = \mathcal{N}(P^\pi) \oplus \mathcal{R}(P^\pi)$ is given by $T = T_1 \oplus N_1$, where T_1 is invertible and N_1 is nilpotent. Throughout the paper we let I represent the identity operator on its domain.

Let P and Q be invertible. The equality

$$(PQ)^{-1} = Q^{-1} P^{-1}$$

is called the reverse order law for the ordinary inverses. It is well known that the reverse order law does not hold for various classes of generalized inverses. Hence, a significant number of papers investigated the sufficient or equivalent conditions such that the reverse order law holds (see [1,3,5,6,9,10,16]). The Moore–Penrose inverse of $T \in \mathcal{B}(\mathcal{H})$ is denoted by $T^+ \in \mathcal{B}(\mathcal{H})$.

$$(PQ)^+ = Q^+ P^+ \ \text{ if and only if } \ \mathcal{R}(P^* PQ) \subset \mathcal{R}(Q), \quad \mathcal{R}(QQ^* P^*) \subset \mathcal{R}(P^*),$$

in the case that P and Q are complex (possibly rectangular) matrices. This result was extended to linear bounded operators on Hilbert spaces by Izumino [10]. Several characterizations of elements P and Q such that $PQ^+ = Q^+ P$ can be found in [1]. In [16] Tian obtained some interesting results concerning the sets of generalized inverses of complex rectangular matrices. In [6] Dragan S. Djordjević and Nebojša Č. Dinčić presented some new results related to the reverse order law for the Moore–Penrose inverse of operators on Hilbert spaces. In [15], Dijana Mosić investigated the reverse order laws for the generalized Drazin inverse in Banach algebras. Similar results related to the Drazin inverse in a ring were presented too.

In this paper we consider some equivalent conditions concerning the reverse order law

$$(PQ)^D = Q^D P^D$$

for the Drazin invertible operators P and Q with partial commutative properties $[P, PQ] = 0$ or $[Q, PQ] = 0$ or $[P, PQQ^D] = 0$ or $[Q, PP^D Q] = 0$, which is useful in several applications, such as in the analysis of Markov chains (see [14]). We specialize in the properties concerning the reverse order law for the Drazin inverses of operators. Some equivalent conditions concerning the reverse order law for the Drazin inverses are given.

2. Some lemmas

In this section we deal with operator matrix

$$\begin{pmatrix} T_1 & T_3 \\ T_4 & T_2 \end{pmatrix}$$

considered on the Hilbert space $\mathcal{H}_1 \oplus \mathcal{H}_2$. The operator T_i acts on the Hilbert space $\mathcal{H}_i, i = 1, 2$, and the operator T_3 (resp. T_4) acts from $\mathcal{H}_2$ to $\mathcal{H}_1$ (resp. from $\mathcal{H}_1$ to $\mathcal{H}_2$). We assume that all entries are