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Global strong solutions to the incompressible Navier–Stokes equations with density-dependent viscosity

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A R T I C L E I N F O

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A B S T R A C T

This paper is concerned with the 3D incompressible Navier–Stokes equations with density-dependent viscosity in a smooth bounded domain. The global well-posedness of strong solutions is established for the case when the bound of density is suitably small, or when the total mass is small with large oscillations. The vacuum is allowed in both cases.

1. Introduction

The present paper is devoted to the study of the incompressible Navier–Stokes equations in $\mathbb{R}^3$:

$$\begin{cases} \rho_t + \mathrm{div}(\rho u) = 0, \\ (\rho u)_t + \mathrm{div}(\rho u \otimes u) + \nabla \Pi = \mathrm{div}(\mu(\rho)\nabla u), \\ \mathrm{div} u = 0, \end{cases} \tag{1.1}$$

with the initial-boundary conditions:

$$(\rho, u)|_{t=0} = (\rho_0, u_0)(x) \quad \text{in } \Omega, \quad u = 0 \quad \text{on } \Omega \times (0, T). \tag{1.2}$$

Here, ρ, u, and Π denote the density, velocity and pressure of the fluid, respectively. $\mu(\rho)$ is the viscosity coefficient assumed to satisfy

$$\mu(\xi) \in C^1[0, \infty) \quad \text{and} \quad 0 < \underline{\mu} \le \mu(\xi) \le \bar{\mu} \quad \text{for} \quad \forall\, \xi \in [0, \infty). \tag{1.3}$$

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The mathematical study of nonhomogeneous incompressible fluids was initiated by Kazhikov, who proved the global existence of weak solutions as well as strong ones when $\mu(\rho)$ is a constant and ρ_0 has a positive lower bound, see [3,4,14]. The unique solvability of (1.1) is first addressed by Ladyzenskaja and Solonnikov [16]. In particular, they proved the global existence of weak solutions and local existence of strong ones of the initial/initial-boundary value problem of (1.1) with large data in dimension $N \geq 2$. It is also well known that the local strong solution is indeed a global one in two dimensions or three dimensions with small data. The global well-posedness for initial data belonging to certain scale invariant spaces, see for example [1,2,8,18].

For the case when the initial data may contain vacuum and the viscosity coefficient $\mu(\rho)$ is still a positive constant, Simon [19] constructed the global weak solutions. By imposing some compatibility condition, Choe–Kim [6] established the local existence of strong solutions. Huang–Wang [11] showed that the local strong solution obtained in [6] is indeed a global one in dimension two. For the three-dimensional case, Kim [15] proved that if $\|\nabla u_0\|_{L^2}$ is sufficiently small, then (1.1) has a unique strong solution, which was generalized by Craig et al. [7] by requiring $\|u_0\|_{\dot{H}^{1/2}}$ small.

When the viscosity coefficient $\mu(\rho)$ depends on ρ, Lions [17] derived the global existence of weak solutions. Later, Desjardins [9] proved the global weak solution with more regularity for the two-dimensional case provided that $\mu(\rho)$ is a small perturbation of a positive constant in L^∞-norm. As for strong solutions away from vacuum, Gui–Zhang [10] obtained the global well-posedness in the case when the initial density ρ_0 is a small perturbation around a positive state in H^s with $s \geq 2$. To overcome the difficulties caused by the presence of vacuum, analogously to [6], Choe–Kim [5] proposed a compatibility condition:

$$-\operatorname{div}(\mu(\rho_0)\nabla u_0) + \nabla\Pi_0 = \rho_0^{1/2} g \quad \text{for some } (\nabla\Pi_0, g) \in L^2, \tag{1.4}$$

and established the local existence of strong solutions. Recently, Huang–Wang [12] obtained the global strong solutions in dimension two, provided $\|\nabla\mu(\rho_0)\|_{L^q}$ $(q > 2)$ is small enough, which had been generalized to the 3D case by Zhang [20] and Huang–Wang [13] when $\|\nabla u_0\|_{L^2}$ is small.

The main result in this paper is to establish global strong solutions under the assumption that the mass or the bound of density is suitably small, which reads as follows.

Theorem 1.1. *For some $q > 3$, assume that the initial data (ρ_0, u_0) satisfies*

$$\begin{cases} 0 \leq \inf\rho_0 \leq \rho_0 \leq \sup\rho_0 \leq \bar{\rho} < \infty, \quad \|\rho_0\|_{L^1} = m, \\ \rho_0 \in W^{1,q}, \quad \|\nabla\mu(\rho_0)\|_{L^q} \leq M, \quad u_0 \in H^1_{0,\sigma} \cap H^2, \end{cases} \tag{1.5}$$

and that the compatibility condition (1.4) holds for some $(\nabla\Pi_0, g) \in L^2$. Then there exist positive constants ϵ and $\widetilde{C}$, depending only on Ω, $\bar{\mu}$, q, $\underline{\mu}$, M, g and $\|\nabla u_0\|_{L^2}$, such that the initial-boundary value problem (1.1)–(1.4) has a global strong solution on $\Omega \times (0, T)$, satisfying

$$\begin{cases} 0 \leq \rho(x,t) \leq \bar{\rho}, \quad \|\nabla\mu(\rho)\|_{L^q} \leq 2M, \quad \forall\ (x,t) \in \Omega \times [0,\infty), \\ (\rho, \mu(\rho)) \in C([0,\infty); W^{1,q}), \quad (\nabla u, \Pi) \in C([0,\infty); H^1) \cap L^1(0,\infty; W^{1,r}), \\ \rho_t \in C([0,\infty); L^q), \quad \sqrt{\rho}u_t \in L^\infty(0,\infty; L^2), \quad u_t \in L^2(0,\infty; H^1_0), \end{cases}$$

for $3 < r < \min\{q, 6\}$, provided

$$\Lambda \triangleq (\bar{\rho}m^2)^{\frac{1}{6}}\bar{\rho}^{\frac{5r-6}{4r}} \max\left\{\bar{\rho}^{\frac{7r+6}{4r}}(1+\bar{\rho})(\bar{\rho}m^2)^{\frac{1}{6}},\ \Lambda_1,\ (\bar{\rho}m^2)^{\frac{1}{6}}\Lambda_2\right\} \leq \epsilon,$$

where