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The complex Brownian motion as a strong limit of processes constructed from a Poisson process

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ABSTRACT

We construct a family of processes, from a single Poisson process, that converges in law to a complex Brownian motion. Moreover, we find realizations of these processes that converge almost surely to the complex Brownian motion, uniformly on the unit time interval. Finally the rate of convergence is derived.

1. Introduction

Kac [17] in 1956 to obtain a solution from a Poisson of the telegraph equation

$$\frac{1}{v}\frac{\partial^2 F}{\partial t^2} = v\frac{\partial^2 F}{\partial x^2} - \frac{2a}{v}\frac{\partial F}{\partial t}, \tag{1}$$

with $a, v > 0$, introduced the processes

$$x(t) = v\int_0^t (-1)^{N_a(r)} dr,$$

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where $N_a = \{N_a(t), t \geq 0\}$ is a Poisson process of intensity a. He noticed that if in equation (1) the parameters a and v tend to infinity with $\frac{2a}{v^2}$ constant and equal to $\frac{1}{D}$, then the equation converges to the heat equation:

$$\frac{1}{D}\frac{\partial F}{\partial t} = \frac{\partial^2 F}{\partial x^2}. \tag{2}$$

Let $x_\varepsilon(t)$ be the processes considered by Kac with $a = \frac{1}{\varepsilon^2}$, $v = \frac{1}{\varepsilon}$. These values satisfy that $\frac{2a}{v^2}$ is constant and $D = \frac{1}{2}$ and we get in (2) an equation whose solution is a standard Brownian motion.

Stroock [20] proved in 1982 that the processes x_ε converge in law to a standard Brownian motion. That is, if we consider (P^ε) the image law of the process x_ε in the Banach space $\mathcal{C}([0,T])$ of continuous functions on $[0,T]$, then (P^ε) converges weakly, when ε tends to zero, towards the Wiener measure. Doing a change of variables, these processes can be written as

$$x_\varepsilon = \left\{ x_\varepsilon(t) := \varepsilon \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(u)} du,\ t \in [0,T] \right\},$$

where $\{N(t), t \geq 0\}$ is a standard Poisson process.

In the mathematical literature we find generalizations with regard to the Stroock result which can be channeled in three directions:

(i) modifying the processes x_ε in order to obtain approximations of other Gaussian processes,
(ii) proving convergence in a stronger sense that the convergence in law in the space of continuous functions,
(iii) weakening the conditions of the approximating processes.

In direction (i), a first generalization is also made by Stroock [20] who modified the processes x_ε to obtain approximations of stochastic differential equations. There are also generalizations, among others, to the fractional Brownian motion (fBm) [18], to a general class of Gaussian processes (that includes fBm) [7], to a fractional stochastic differential equation [3], to the stochastic heat equation driven by Gaussian white noise [2] or to the Stratonovich heat equation [8].

On the other hand, there is some literature where the authors present realizations of the processes that converge almost surely, uniformly on the unit time interval. These processes are usually called uniform transport processes. Since the approximations always start increasing, a modification of the processes as

$$\tilde{x}_\varepsilon(t) = \varepsilon(-1)^A \int_0^{\frac{t}{\varepsilon^2}} (-1)^{N(u)} du,$$

has to be considered where $A \sim \text{Bernoulli}\left(\frac{1}{2}\right)$ independent of the Poisson process N.

Griego, Heath and Ruiz-Moncayo [16] showed that these processes converge strongly and uniformly on bounded time intervals to Brownian motion. In [14] Gorostiza and Griego extended the result to diffusions. Again Gorostiza and Griego [15] and Csörgő and Horváth [6] obtained a rate of convergence. More precisely, in [15] it is proved that there exist versions of the transport processes $\tilde{x}_\varepsilon$ on the same probability space as a given Brownian motion $(W(t))_{t\geq 0}$ such that, for each $q > 0$,

$$P\left(\sup_{a\leq t\leq b} |W(t) - \tilde{x}_\varepsilon(t)| > C\varepsilon \left(\log \frac{1}{\varepsilon^2}\right)^{\frac{5}{2}}\right) = o\left(\varepsilon^{2q}\right),$$