J. Math. Anal. Appl. 444 (2016) 745–754

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On reciprocal transformation of a 3-component Camassa–Holm type system

Nianhua Li

School of Mathematics, Huaqiao University, Quanzhou, Fujian 362021, People's Republic of China

ARTICLE INFO

Article history:
Received 7 March 2016
Available online 7 July 2016
Submitted by J. Lenells

Keywords:
Camassa–Holm type equation
Reciprocal transformation
Hamiltonian structure

ABSTRACT

We construct a reciprocal transformation to connect a 3-component Camassa–Holm type system proposed by Geng and Xue with the first negative flow in a generalized MKdV hierarchy. We discuss the Hamiltonian pair and infinitely many conserved quantities for the generalized MKdV hierarchy.

1. Introduction

Since the celebrated Camassa–Holm (CH) equation was derived as a model for unidirectional motion of dispersive shallow-water by Camassa and Holm in 1993 [4,5], integrable PDEs admitting peakons (CH type equations) have attracted much attention in recent years. Two of the most famous equations are the CH equation and the Degasperis–Procesi (DP) equation, which are contained in the following one parameter family of PDEs [8]:

$$m_t + um_x + bu_x m = 0, \quad m = u - u_{xx}, \tag{1}$$

where b is an arbitrary constant. When $b = 2$, the equation is the CH, which is completely integrable with a Lax pair and a bi-Hamiltonian structure [4,5]. The CH equation is solvable by inverse scattering transform [6] and possesses algebro-geometric solutions [16]. And explicit formulas for the multi-peakon solutions of the equation are studied by the inverse spectral methods [2,3]. Moreover, it is connected to the negative KdV equation by a transformation of reciprocal type [11,17,21]. When $b = 3$, the system (1) is just the DP equation, which is discovered by the method of asymptotic integrability to isolate integrable third-order equations [7]. Indeed it can be obtained by the theory of shallow water [9]. The DP equation is connected to a negative flow in the Kaup–Kupershmidt hierarchy via a reciprocal transformation. With the aid of the

E-mail address: linianh@hqu.edu.cn.

http://dx.doi.org/10.1016/j.jmaa.2016.06.072

transformation, a Lax pair for the DP equation is constructed. Besides the bi-Hamiltonian structure and the integrability of the finite-dimensional peakon dynamics for the equation are discussed [8]. Furthermore, explicit formulas for the multi-peakon solutions of the DP equation are researched in Refs. [28,29].

In the past two decades, many other CH type equations are proposed and studied (see e.g. [13,18,26] and references). For example, the Novikov equation [31]

$$m_t + u^2 m_x + 3uu_x m = 0, \quad m = u - u_{xx} \tag{2}$$

is discovered by the method of symmetry classification. The bi-Hamiltonian structure and a Lax pair for the Novikov equation are given and there is a reciprocal transformation to connect it with a negative flow in the Sawada–Kotera hierarchy [19]. In addition, explicit formulas for multi-peakon solutions of it are obtained in Ref. [20]. The Geng–Xue equation [14]

$$m_t + 3u_x vm + uvm_x = 0, \tag{3}$$

$$n_t + 3uv_x n + uvn_x = 0, \tag{4}$$

$$m = u - u_{xx}, \quad n = v - v_{xx} \tag{5}$$

is proposed as a generalization of the Novikov equation. It is also completely integrable with a Lax pair and associated bi-Hamiltonian structure as well as infinitely many conserved quantities [14,23]. Furthermore, a reciprocal transformation is constructed to connect it with the first negative flow in the modified Boussinesq hierarchy [25], and explicit formulas for the multi-peakon of the equation are considered in Ref. [30].

The subject of this paper is a 3-component CH type system proposed by Geng and Xue [15], i.e.

$$\begin{aligned} u_t &= -vp_x + u_x q + \frac{3}{2}uq_x - \frac{3}{2}u(p_x r_x - pr), \\ v_t &= 2vq_x + v_x q, \\ w_t &= vr_x + w_x q + \frac{3}{2}wq_x + \frac{3}{2}w(p_x r_x - pr), \end{aligned} \tag{6}$$

where

$$\begin{aligned} u &= p - p_{xx}, \\ v &= \frac{1}{2}(q_{xx} - 4q + p_{xx}r_x - r_{xx}p_x + 3p_x r - 3pr_x), \\ w &= r_{xx} - r, \end{aligned}$$

which admits the following spectral problem

$$\psi_x = \begin{pmatrix} 0 & 1 & 0 \\ 1 + \lambda v & 0 & u \\ \lambda w & 0 & 0 \end{pmatrix} \psi. \tag{7}$$

The bi-Hamiltonian structure as well as infinite many conserved quantities and the dynamical system of N-peakon solutions for this system are obtained in Refs. [15,24]. It is interesting that the spectral problem (7) for this 3-component model may be reduced to that of the CH equation and that of the Geng–Xue equation, the same is true for the associated bi-Hamiltonian structures. Both of these will be done below. There exists another reason why we are intrigued about reciprocal transformations of the system (6). Note that the spectral problem (7) is gauge equivalent to that of another 3-component CH type system constructed by