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A mathematical justification of a thin film approximation for the flow down an inclined plane

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ABSTRACT

We consider a two-dimensional motion of a thin film flowing down an inclined plane under the influence of the gravity and the surface tension. In order to investigate the stability of such flow, we often use a thin film approximation, which is an approximation obtained by the perturbation expansion with respect to the aspect ratio of the film. The famous examples of the approximate equations are the Burgers equation, Kuramoto–Sivashinsky equation, KdV–Burgers equation, KdV–Kuramoto–Sivashinsky equation, and so on. In this paper, we give a mathematically rigorous justification of a thin film approximation by establishing an error estimate between the solution of the Navier–Stokes equations and those of approximate equations.

1. Introduction

In this paper, we consider a two-dimensional motion of a liquid film of a viscous and incompressible fluid flowing down an inclined plane under the influence of the gravity and the surface tension on the interface. The motion can be mathematically formulated as a free boundary problem for the incompressible Navier–Stokes equations. We assume that the domain $\Omega(t)$ occupied by the liquid at time $t \geq 0$, the liquid surface $\Gamma(t)$, and the rigid plane Σ are of the form

$$\begin{cases} \Omega(t) = \{(x,y) \in \mathbb{R}^2 \mid 0 < y < h_0 + \eta(x,t)\}, \\ \Gamma(t) = \{(x,y) \in \mathbb{R}^2 \mid y = h_0 + \eta(x,t)\}, \\ \Sigma = \{(x,y) \in \mathbb{R}^2 \mid y = 0\}, \end{cases}$$

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where h_0 is the mean thickness of the liquid film and $\eta(x,t)$ is the amplitude of the liquid surface. Here we choose a coordinate system (x,y) so that x axis is pointed to the streamwise direction and y axis is normal to the plane. We consider fluctuations of the Nusselt flat film solution, which is the stationary laminar flow given by

$$\eta_1 = 0, \quad u_1 = (\rho g \sin\alpha/2\mu)(2h_0 y - y^2), \quad v_1 = 0, \quad p_1 = p_0 - \rho g \cos\alpha(y - h_0), \tag{1.1}$$

where ρ is a constant density of the liquid, g is the acceleration of the gravity, α is the angle of inclination, μ is the shear viscosity coefficient, and p_0 is an atmospheric pressure. Throughout this paper, we assume that the flow is l_0-periodic in the streamwise direction x. Rescaling the independent and dependent variables by using h_0, l_0, the typical amplitude of the liquid surface a_0, $U_0 = \rho g h_0^2 \sin\alpha/2\mu$, and $P_0 = \rho g h_0 \sin\alpha$, the equations are written in the non-dimensional form

$$\begin{cases} \delta \boldsymbol{u}_t + \big((\bar{\boldsymbol{u}} + \varepsilon \boldsymbol{u})\cdot\nabla_\delta\big)\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla_\delta)\bar{\boldsymbol{u}} + \dfrac{2}{\mathrm{R}}\nabla_\delta p - \dfrac{1}{\mathrm{R}}\Delta_\delta \boldsymbol{u} = \boldsymbol{0} & \text{in} \quad \Omega_\varepsilon(t),\ t>0, \\ \nabla_\delta\cdot\boldsymbol{u} = 0 & \text{in} \quad \Omega_\varepsilon(t),\ t>0, \end{cases} \tag{1.2}$$

$$\begin{cases} \big(\boldsymbol{D}_\delta(\varepsilon\boldsymbol{u} + \bar{\boldsymbol{u}}) - \varepsilon p \boldsymbol{I}\big)\boldsymbol{n} & \\ \quad = \Big(-\dfrac{1}{\tan\alpha}\varepsilon\eta + \dfrac{\delta^2 \mathrm{W}}{\sin\alpha}\dfrac{\varepsilon\eta_{xx}}{(1+(\varepsilon\delta\eta_x)^2)^{\frac{3}{2}}}\Big)\boldsymbol{n} & \text{on} \quad \Gamma_\varepsilon(t),\ t>0, \\ \eta_t + \big(1-(\varepsilon\eta)^2 + \varepsilon u\big)\eta_x - v = 0 & \text{on} \quad \Gamma_\varepsilon(t),\ t>0, \end{cases} \tag{1.3}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on} \quad \Sigma,\ t>0. \tag{1.4}$$

Here, $\delta, \varepsilon, \mathrm{R}$, and W are non-dimensional parameters defined by

$$\delta = \frac{h_0}{l_0}, \quad \varepsilon = \frac{a_0}{h_0}, \quad \mathrm{R} = \frac{\rho U_0 h_0}{\mu}, \quad \mathrm{W} = \frac{\sigma}{\rho g h_0^2},$$

where σ is the surface tension coefficient. Note that δ is the aspect ratio of the film, ε represents the magnitude of nonlinearity, R is the Reynolds number, and W is the Weber number. Moreover, we used notations $\boldsymbol{u} = (u, \delta v)^{\mathrm{T}}$, $\bar{\boldsymbol{u}} = (\bar{u}, 0)^{\mathrm{T}}$, $\bar{u} = 2y - y^2$, $\nabla_\delta = (\delta\partial_x, \partial_y)^{\mathrm{T}}$, $\Delta_\delta = \nabla_\delta\cdot\nabla_\delta$, $\boldsymbol{D}_\delta \boldsymbol{f} = \frac{1}{2}\{\nabla_\delta(\boldsymbol{f}^{\mathrm{T}}) + \big(\nabla_\delta(\boldsymbol{f}^{\mathrm{T}})\big)^{\mathrm{T}}\}$, and $\boldsymbol{n} = (-\varepsilon\delta\eta_x, 1)^{\mathrm{T}}$. In this scaling, the liquid domain $\Omega_\varepsilon(t)$ and the liquid surface $\Gamma_\varepsilon(t)$ are of the form

$$\begin{cases} \Omega_\varepsilon(t) = \{(x,y)\in\mathbb{R}^2 \mid 0 < y < 1 + \varepsilon\eta(x,t)\}, \\ \Gamma_\varepsilon(t) = \{(x,y)\in\mathbb{R}^2 \mid y = 1 + \varepsilon\eta(x,t)\}. \end{cases}$$

Concerning a mathematical analysis of the problem in the case of $\delta = \varepsilon = 1$, Teramoto [15] showed that the initial value problem to the Navier–Stokes equations (1.2)–(1.4) has a unique solution globally in time under the assumptions that the Reynolds number and the initial data are sufficiently small. Nishida, Teramoto, and Win [11] showed the exponential stability of the Nusselt flat film solution under the assumptions that the angle of inclination is sufficiently small and $x \in \mathbb{T}$ in addition to the assumptions in [15]. Furthermore, Uecker [16] studied the asymptotic behavior for $t \to \infty$ of the solution in the case of $x \in \mathbb{R}$ and showed that the perturbations of the Nusselt flat film solution decay like the self-similar solution of the Burgers equation under the assumptions that the initial data are sufficiently small and $\mathrm{R} < \mathrm{R}_c$. Here, $\mathrm{R}_c = \frac{4}{5}\frac{1}{\tan\alpha}$ is the critical Reynolds number given by Benjamin [2]. On the other hand, Ueno, Shiraishi, and Iguchi [17] derived a uniform estimate for the solution of (1.2)–(1.4) with respect to δ when the Reynolds number, the angle of inclination, and the initial data are sufficiently small.