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The structure of solutions near a sonic line in gas dynamics via the pressure gradient equation

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ABSTRACT

A key issue in gas dynamics in two space dimensions is the regularity of solutions near a sonic curve. We build a large class of regular solutions with given boundary conditions on the sonic line. The modeling equation is the pressure gradient equation, which is the same as Euler system when the parameters of the gas are pushed to certain extreme. We use a novel set of coordinates, involving both the space–time and state variables, to split regular terms from singular terms in the analysis.

1. Introduction

The Euler system models the motion of compressible ideal fluids and has been discussed a lot theoretically, experimentally and numerically in the literature (see [5,7,8]). The pressure gradient (PG) system is derived out of the Euler system either through flux splitting [15] or asymptotic expansion [26,28]. The two dimensional PG system takes the form

$$\begin{cases} u_t + p_x = 0, \\ v_t + p_y = 0, \\ p_t + pu_x + pv_y = 0, \end{cases}$$

where (u, v) is velocity and p is pressure. One feature of this system is that pressure can be decoupled from u, v to form its own second order quasilinear differential equation

$$(\frac{p_t}{p})_t - p_{xx} - p_{yy} = 0. \tag{1.1}$$

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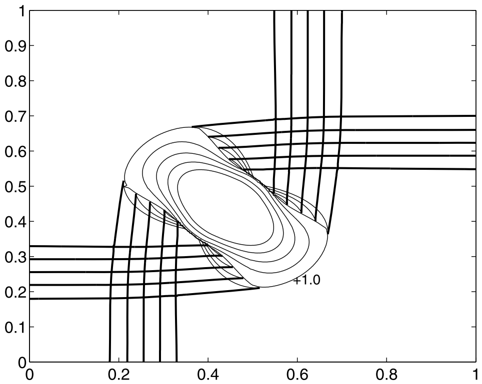

Fig. 1. Interaction of two forward and two backward rarefaction waves for Euler with $\gamma = 1.4$, pressure $p_1 = 0.444$, density $\rho_1 = 1.0$, $\rho_2 = 0.5197$, velocity $u_1 = v_1 = 0.00$ at time T = 0.25. The contour curves are pseudo-March lines where the outmost one marked with $+1.0$ is the sonic curve. The bold curves from the boundaries and light short ones are all characteristics. The four regions where the bold and the short light characteristics overlap are semi-hyperbolic regions. (Courtesy of Glimm et al. [10].)

The PG system is considered as a useful simplified model for the Euler equations, since interesting observations sometimes were first found in this model and later recovered for the Euler system. Furthermore, the PG system is easier to handle technically.

The well-posedness of the general Cauchy problem or the two dimensional Riemann problem, where the initial data is a constant along each ray through the origin on the physical (x, y) plane, remains a largely open question. There is work though in [6,3,2,27] for constructing the transonic shock solutions in particular situations for the Euler system and other models arising from gas dynamics. We refer to Numerical simulations of the Riemann problem in [11,12,16,4].

The four-wave Riemann problem refers to the initial data, where it is a constant in each quadrant and adjacent state is connected by a single wave. It is a special case of the 2-D Riemann problem. Self-similar solutions depending only on $(\xi = \frac{x}{t}, \eta = \frac{y}{t})$ are expected. And the equation (1.1) is turned into a new form

$$(p - \xi^2)p_{\xi\xi} - 2\xi\eta p_{\xi\eta} + (p - \eta^2)p_{\eta\eta} + \frac{1}{p}(\xi p_\xi + \eta p_\eta)^2 - 2(\xi p_\xi + \eta p_\eta) = 0. \tag{1.2}$$

The PG equation (1.2) changes type as the Euler equations in the self-similar plane. A *sonic curve* is where the equation changes type from hyperbolic to elliptic. A conjecture was proposed in [22] with supportive numerical results in [19,16,22] that the solutions of the 4-wave problem have 19 different configurations for polytropic gas modeled by Euler equations and 12 genuinely different configurations for the PG system.

The existence of solution in the elliptic region with data assigned on the sonic curve was proved in [25] for the PG system. Various wave interactions in the hyperbolic region were analyzed in [9,13,14,1,27]. A global classic solution was constructed in [17] to the interaction of four orthogonal planer rarefaction waves with two axes of symmetry for the Euler system. The strengths of the waves were chosen to be large to avoid the occurrence of sonic points and the solution was hyperbolic all the way to the vacuum. We remark that without the restriction on the wave strength, a subsonic domain would be involved in the self-similar plane and the solution will be transonic, which is more interesting yet much more challenging to study.

Another solution for the Euler system related with the transonic phenomenon was constructed in [18] with data assigned on two characteristics, which intersect the sonic curve in two distinguishable points. The solution was proved to exist in a region bounded by the two characteristics and part of the sonic curve. Such a solution is called a *semi-hyperbolic patch* since one family of characteristics emanating from the sonic boundary would form a transonic shock. It was also observed in the numerical simulation [10] in the region covered by overlapping bold and light curves (see Fig. 1). The characteristics are not tangential to the sonic curve when vanishing in the semi-hyperbolic patch (see Fig. 1), which behaves differently as in the