J. Math. Anal. Appl. 443 (2016) 81–91

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A reflection result for harmonic functions which vanish on a cylindrical surface

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A R T I C L E I N F O

Article history:
Received 5 February 2016
Available online 11 May 2016
Submitted by T. Ransford

Keywords:
Harmonic continuation
Green function
Cylindrical harmonics

A B S T R A C T

Suppose that a harmonic function h on a finite cylinder U vanishes on the curved part A of the boundary. It was recently shown that h then has a harmonic continuation to the infinite strip bounded by the hyperplanes containing the flat parts of the boundary. This paper examines what can be said if the above function h is merely harmonic near A (and inside U). It is shown that h then has a harmonic extension to a larger domain formed by radial reflection.

1. Introduction

Let $N \geq 3$ and $a > 0$, and let B' denote the open unit ball in $\mathbb{R}^{N-1}$. The following harmonic extension result for cylinders was recently established in [4].

Theorem 1. *Any harmonic function on the finite cylinder $B' \times (-a, a)$ which continuously vanishes on $\partial B' \times (-a, a)$ has a harmonic extension to $\mathbb{R}^{N-1} \times (-a, a)$.*

In the case where $N = 2$, Theorem 1 is easily verified by repeated application of the Schwarz reflection principle. In higher dimensions the result was proved by a detailed analysis of series expansions involving Bessel functions. It is natural to ask whether some sort of extension result still holds when the given harmonic function is merely defined near the curved boundary (and inside the cylinder). The corresponding assertion certainly holds when $N = 2$, as can again be seen by Schwarz reflection. However, there is an obstacle to this approach in higher dimensions, since Ebenfelt and Khavinson [3] (see also Chapter 10 of [5]) have shown that a point-to-point reflection law in $\mathbb{R}^3$ can only hold for planar or spherical surfaces. Nevertheless, as will be seen below, it is still possible to establish harmonic extension to a "radial reflection" of the original domain.

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http://dx.doi.org/10.1016/j.jmaa.2016.05.007
0022-247X/

Let (x', x_N) denote a typical point of $\mathbb{R}^{N-1} \times \mathbb{R}$ and $\|x'\|$ denote the Euclidean norm of x'. Our main result is as follows.

Theorem 2. *Let $\phi : (-a, a) \to [0, 1)$ be upper semicontinuous. Then any harmonic function on the domain*

$$\{(x', x_N) : |x_N| < a \quad and \quad \phi(x_N) < \|x'\| < 1\} \tag{1}$$

which continuously vanishes on $\partial B' \times (-a, a)$ has a harmonic extension to the domain

$$\{(x', x_N) : |x_N| < a \quad and \quad \phi(x_N) < \|x'\| < 2 - \phi(x_N)\}. \tag{2}$$

The sharpness of the upper bound $2 - \phi(x_N)$ in (2) is demonstrated by the example below.

Example. Let $N = 4$ and $\phi : [-a, a] \to [0, 1)$ be continuous. Then there is a harmonic function on the domain (1) which continuously vanishes on $\partial B' \times (-a, a)$ and does not have a harmonic extension beyond the domain (2).

To see this, let $\omega = \{(s, t) \in \mathbb{R}^2 : \phi(t) < s < 1, |t| < a\}$, let u be the logarithmic potential of a measure comprising a dense sequence of point masses in the set $\{(s, t) \in \partial\omega : s \neq 1\}$, and let $v(s, t) = u(s, t) - u(2 - s, t)$. Thus v is harmonic on the domain $\{(s, t) : \phi(t) < s < 2 - \phi(t), |t| < a\}$, is unbounded near each boundary point, and vanishes on $\{1\} \times (-a, a)$. The function

$$(x', x_4) \mapsto \|x'\|^{-1} v(\|x'\|, x_4) \quad (x' \in \mathbb{R}^3 \backslash \{0\})$$

is now easily seen (by computation of the Laplacian) to be harmonic on the domain (2) and to vanish on $\partial B' \times (-a, a)$, yet it does not have a harmonic extension beyond (2).

Remark. The special case of Theorem 2 where the harmonic function is of the form $f(\|x'\|, x_N)$ follows easily from known reflection results in the plane (see Lewy [7]), since $\Delta f + (N-2)s^{-1}\partial f/\partial s = 0$ on the domain $\{(s, t) : |t| < a, \phi(t) < s < 1\}$ and $f = 0$ on the boundary line segment $\{1\} \times (-a, a)$. There are even explicit formulae for the extension in this case: see Savina [8]. (We are grateful to Dima Khavinson for these references.)

The proof of Theorem 2 will combine results from [4] with several additional arguments.

2. Preparatory material

Let J_ν and Y_ν denote the usual Bessel functions of order $\nu \geq 0$, of the first and second kinds (see Watson [11]). Further, let $(j_{\nu,m})_{m\geq 1}$ denote the sequence of positive zeros of J_ν, arranged in increasing order. We collect below a few facts that we will need.

Lemma 3.

(i) $\dfrac{d}{dz}\dfrac{J_\nu(z)}{z^\nu} = -\dfrac{J_{\nu+1}(z)}{z^\nu}$.
(ii) $|J_\nu(t)| \leq 1 \ (t > 0)$.
(iii) $j_{\nu,m} \geq (m + 3/4)\pi + \nu \ (m \geq 1)$.
(iv) *If $y(t) = \sqrt{t}J_\nu(kt)$, where k is a non-zero constant, then*

$$\frac{d^2y}{dt^2} + \left(k^2 + \frac{\frac{1}{4} - \nu^2}{t^2}\right) y = 0 \quad (t > 0).$$