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The atomic decomposition of weighted Hardy spaces associated to self-adjoint operators on product spaces ☆

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ABSTRACT

Let L be a non-negative self-adjoint operator acting on $L^2(\mathbb{R}^n)$ satisfying a pointwise Gaussian estimate for its heat kernel. Let w be an A_p weight on $\mathbb{R}^n \times \mathbb{R}^n$, $1 < p < \infty$. In this article we obtain a weighted q-atomic decomposition with $q \geq p$ for the weighted Hardy space $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ associated to L.

1. Introduction

The theory of Hardy spaces on product domains was initiated by Gundy and Stein [27]. The atomic decompositions for Hardy spaces on product domains were obtained by Chang and Fefferman [11,13]. Later, Fefferman [24], Krug [33], Sato [39] and others established the weighted theory of the classical Hardy spaces on product domains.

Recently, there are some important situations in which the theory of classical Hardy spaces is not applicable. Some interesting operators were found to be out of Calderón–Zygmund class. They are strong type $(2, 2)$, but their kernels do not possess regularity properties such as the Hörmander condition. They are found to be strong type (p, p) for a range of values of p different from $1 < p < 2$. They also are unbounded in classical Hardy spaces. See [20,2,29,5,6,4]. In order to handle such beyond Calderón–Zygmund operators, many researchers study Hardy spaces that are adapted to a linear operator L, in much the same way that the classical Hardy spaces are adapted to the Laplacian.

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We first recall some results of the one-parameter theory. First Auscher, Duong and McIntosh [3], and then Duong and Yan [23], introduced Hardy spaces adapted to an operator L whose heat kernel enjoys a pointwise Gaussian upper bound. In [7] and in [29], the authors treated Hardy spaces adapted, respectively, to the Hodge Laplacian on a Riemannian manifold with doubling measure, or to a second order divergence form elliptic operator on $\mathbb{R}^n$ with complex coefficients, in which settings the pointwise Gaussian upper bound of heat kernel may fail. Much of the theory was carried out in [7,29,30,19] using only Davies–Gaffney type estimates in place of pointwise kernel bounds. After that, in [28], the authors consider a non-negative self-adjoint operator L satisfying Davies–Gaffney bounds on L^2. They developed a theory of Hardy (and BMO) spaces associated to L, including an atomic decomposition, square function characterization. For more study of the theory of function spaces associated to operators, we refer to [22,9,32,41].

Most recently, Song and Yan presented a theory of the weighted Hardy spaces associated to operators [42]. Roughly speaking, for the non-negative self-adjoint operators L whose heat kernel enjoys a pointwise Gaussian upper bound, and for $w \in A_1 \cap \mathrm{RH}_2$, they introduced weighted Hardy spaces $H^1_{L,w}(\mathbb{R}^n)$ associated to L in terms of the area function characterization, and proved their atomic decomposition. Here, A_1 is Muckenhoupt weight, and RH_2 is the reverse Hölder class. Latter, the authors of this paper weakened the conditions on weights of the weighted Hardy spaces $H^1_{L,w}(\mathbb{R}^n)$ [35] and obtained another atomic decomposition (different from that of [42]) for $H^1_{L,w}(\mathbb{R}^n)$. In [35], the condition $w \in A_1 \cap \mathrm{RH}_2$ was weakened to $w \in A_p$, $1 < p < \infty$. The main tools were Whitney decomposition, and an integral inequality on set of points of density which was proved in [15]. For the theory of weighted Hardy spaces associated with a second order divergence form elliptic operator, we refer the reader to a series of papers [36,37].

This paper concerns the weighted Hardy spaces associated to operators on product spaces. Let L be a non-negative self-adjoint operator acting on $L^2(\mathbb{R}^n)$ satisfying a pointwise Gaussian estimate for its heat kernel. Let w be an A_p weight on $\mathbb{R}^n \times \mathbb{R}^n$ (see Section 2 for its definition), where $1 < p < \infty$. The main purpose of this paper is to introduce a weighted Hardy space $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ and prove a weighted atomic decomposition for $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$. See Theorem 4.1 and Theorem 4.4, which are our main results of this paper.

In comparison with the one-parameter case, the situation becomes considerably complicated. As pointed by [10], the product atoms should be supported in open sets rather than rectangles, which lead to many difficulties. Since there is no Whitney decomposition in product domains, the argument of [35] does not work.

Two important tools will play crucial roles in our paper. The first one is a weighted version of Journé's covering lemma, which is a good substitute in product domain for Whitney decomposition. It will be used to prove (i) of Theorem 4.1. The second one is Lemma 4.3, which is a modified version of an estimate in [12] and will be used to prove (ii) of Theorem 4.1. In order to prove Lemma 4.3, we also need to combine the idea of [12,24] with the semigroup property.

The layout of the article is as follows. In section 2, we introduce some notation, basic assumption and state some preliminary results. In Section 3, we define the weighted Hardy space $H^1_{L,w}(\mathbb{R}^n \times \mathbb{R}^n)$ associated to a non-negative self-adjoint operator with Gaussian upper bounds on its heat kernel. We also define the weighted product atoms associated to the operator. In Section 4, we state and prove the atomic characterizations of weighted product Hardy spaces, which is the main result of this paper.

2. Notation and preliminaries

2.1. Notation

For $1 \le p \le \infty$, we will denote p' the adjoint number of p, i.e. $1/p + 1/p' = 1$. If Q is cube, then $\ell(Q)$ will denote the side length of Q. When B is a ball or rectangle, and a is a positive number, we shall use aB to denote the a-fold dilate of B concentric with B. We also denote $d(E, F)$ the distance of two sets E and F.