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Critical exponent for evolution equations in modulation spaces ☆

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A B S T R A C T

In this paper, we propose a method to find the critical exponent for certain evolution equations in modulation spaces. We define an index $\sigma(s, q)$, and use it to determine the critical exponent of the fractional heat equation as an example. We prove that when $\sigma(s, q)$ is greater than the critical exponent, this equation is locally well posed in the space $C(0, T; M^s_{p,q})$; and when $\sigma(s, q)$ is less than the critical exponent, this equation is ill-posed in the space $C(0, T; M^s_{2,q})$. Our method may further be applied to some other evolution equations.

1. Introduction and main results

As we all know, many evolution equations have their critical exponents on either Sobolev spaces or Besov spaces, or both. For example, the critical exponent of nonlinear Schrödinger equation (NLS) in Besov spaces $\dot{B}^s_{p,2}$ is $\frac{n}{p} - \frac{2}{k-1}$ when the nonlinear term is $|u|^{k-1}u$. Cazenave and Weissler [4] showed that NLS is locally well-posed in $C([-T, T]; \dot{H}^s)$ when $s \geq 0$ and $s > \frac{n}{2} - \frac{2}{k-1}$. In [6], Christ, Colliander and Tao proved that when $s < \max\{0, \frac{n}{2} - \frac{2}{k-1}\}$, NLS is ill-posed in $\dot{H}^s$. In [11], Miao, Xu and Zhao proved similar results for the nonlinear Hartree equation. We observe that both works in [6] and [11] are heavily based on the scaling invariance of the work space. On the other hand, the modulation space $M^s_{p,q}$ is lack of the scaling property, although this space emerges in recent years and plays a significant role in the study of certain nonlinear evolution equations. (We will describe more details of the modulation space in the following content.) Since we are not able to find in literature any study on critical exponent for evolution equation in the modulation space, the aim of this paper is to propose a different method from [6] and [11] to find the critical exponents. Particularly we find the critical exponent for the fractional heat equation in the modulation space, without the scaling invariance. This exponent satisfies the well and ill posedness property on the modulation space, which is quite similar to that for NLS in the Sobolev space.

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Modulation spaces were introduced by Feichtinger in [7] to measure smoothness of a function or distribution in a way different from L^p spaces, and they are now recognized as a useful tool for studying pseudo-differential operators (see [2,5,13,15,16]). The original definition of the modulation space is based on the short-time Fourier transform and window function. In [19], Wang and Hudizk gave an equivalent definition of the discrete version on modulation spaces by the frequency-uniform–decomposition. With this discrete version, they are able to find global solutions for nonlinear Schrödinger equation and nonlinear Klein–Gordon equation in lower regularity spaces. After then, there are many studies on nonlinear PDEs in modulation spaces followed their work. Below we list some of them, among many others. In [9], Guo and Chen proved the Stricharz estimates on α-modulation spaces. For well-posedness in modulation space, Wang, Zhao, and Guo [20] studied the local solution for nonlinear Schrödinger equation and Navier–Stokes equations. In [18], Wang and Huang studied the local and global solutions for generalized KdV equations, Benjamin–Ono and Schrödinger equations. In [14], Ruzhansky, Sugimoto and Wang stated some new progress and open questions in modulation spaces. Also, for the ill posedness in modulation spaces, Iwabuchi studied well and ill posedness for Navier–Stokes equations and heat equations (see [10]). Iwabuchi's result can be stated in the following theorem:

Theorem A. *[10]: When $s - \frac{n}{q'} > -\frac{2}{k-1}$, the Heat equation*

$$(H) \qquad u(t) = e^{t\Delta}u_0 + \int_0^t e^{(t-\tau)\Delta}u^k d\tau$$

is locally well-posed in $C([0, T, M^s_{p,q}])$. When $s < -\frac{2}{k}$ or $s - \frac{n}{q'} < -\frac{n+2}{k}$, equation (H) is ill-posed in $C([0, T, M^s_{2,q}])$.

Since Iwabuchi's result is not a sharp one, a natural question is if there are some critical exponents for this equation in modulation spaces based on the well and ill posedness. In this paper, we will answer this question.

First, we recall some important properties of Besov spaces [8]. The first one is a Sobolev-type embedding that says $B^{s_1}_{p_1,q} \subset B^{s_2}_{p_2,q}$ if and only if

$$s_2 \leq s_1 \quad and \quad s_1 - \frac{n}{p_1} \geq s_2 - \frac{n}{p_2}.$$

The second one says that the Besov space $B^s_{p,q}$ forms a multiplication algebra if $s - \frac{n}{p} > 0$. By comparing these properties to the Proposition 2 and Proposition 3 in Section 2, we observe that the index $s - \frac{n}{p}$ in the Besov space is an analog of the index $s - \frac{n}{q'}$ in the modulation space. Motivated by such an observation, heuristically, we may use the index $s - \frac{n}{q'}$ to describe the critical exponent in the modulation space. Of course, this heuristic idea will be technically supported in our following discussion. For convenience in the discussion, we denote $\sigma(s, q) = s - \frac{n}{q'}$, and use the inequality

$$A(u, v, w...) \lesssim B(u, v, w...)$$

to mean that there is a positive number C independent of all main variables $u, v, w...$, for which $A(u, v, w...) \leq CB(u, v, w...)$.

Now we state a general theorem for well posedness.

Theorem 1. *Let $U(t)$ be the dispersive semigroup:*

$$U(t) := \mathcal{F}^{-1}e^{tp(\xi)}\mathcal{F}$$