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The Cauchy problem for a class of parabolic equations in weighted variable Sobolev spaces: Existence and asymptotic behavior

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ABSTRACT

The paper addresses the questions of existence and asymptotic behavior of solutions to the Cauchy problem for the equation

$$u_t - \operatorname{div}(D(x)|\nabla u|^{p(x)-2}\nabla u) + A(x)|u|^{q(x)-2}u = f(x,t,u).$$

The coefficients D, A are nonnegative functions which may vanish on a set of zero measure in $\mathbb{R}^n$, and $A(x) \to \infty$ as $|x| \to \infty$, $f(x,t,u)$ is globally Lipschitz with respect to u. The exponents $p, q : \mathbb{R}^n \mapsto (1,\infty)$ are given measurable functions. We prove that the problem admits at least one weak solution in a weighted Sobolev space with variable exponents, provided that $p^- = \operatorname{ess\,inf}_{\mathbb{R}^n} p(x) > \max\left\{\frac{2n}{n+2}, 1\right\}$, $q^- = \operatorname{ess\,inf}_{\mathbb{R}^n} q(x) > 2$, $A^{-\frac{2}{q(x)-2}} \in L^1(\mathbb{R}^n)$ and $D^{-\frac{s}{p(x)-s}} \in L^1(B_{R_1}(0))$ with constants $\max\left\{1, \frac{2n}{n+2}\right\} < s < \min\{p^-, q^-\}$ and $R_1 > 0$. In the case $p^- > 2$, $q(x) = p(x)$ a.e. in $\mathbb{R}^n$, and $f \equiv f(u)$, there exists a unique strong solution and the problem has a global attractor in $L^2(\mathbb{R}^n)$.

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1. Introduction

The paper is devoted to study the Cauchy problem for the class of quasilinear degenerate parabolic equations with variable nonlinearity and unbounded coefficients. We consider the problem

$$\begin{cases} u_t - \operatorname{div} \mathcal{A}(x, \nabla u) + \mathcal{B}(x, t, u) = 0 \quad \text{in } S_T := \mathbb{R}^n \times (0, T], \\ u(x, 0) = u_0(x) \text{ in } \mathbb{R}^n, \end{cases} \tag{1.1}$$

where the nonlinear terms have the form

$$\mathcal{A}(x, \nabla u) = D(x)|\nabla u|^{p(x)-2}\nabla u, \qquad \mathcal{B}(x, t, u) = A(x)|u|^{q(x)-2}u - f(x, t, u)$$

with given measurable on $\mathbb{R}^n$ exponents $p(x)$ and $q(x)$, and a given function f. It is assumed throughout the paper that

$$\begin{aligned} 1 < p^- &:= \operatorname{ess\,inf}_{\mathbb{R}^n} p(x) \le p(x) \le p^+ := \operatorname{ess\,sup}_{\mathbb{R}^n} p(x) < \infty, \\ 1 < q^- &:= \operatorname{ess\,inf}_{\mathbb{R}^n} q(x) \le q(x) \le q^+ := \operatorname{ess\,sup}_{\mathbb{R}^n} q(x) < \infty. \end{aligned} \tag{1.2}$$

The given coefficients $D(x)$, $A(x)$ are subject to the following restrictions:

H1) $D(x), A(x) \in L^\infty_{loc}(\mathbb{R}^n)$, $\quad D(x), A(x)$ are nonnegative a.e. in $\mathbb{R}^n$,
H2) there exist constants $R_1 > 0$ and $\beta > 0$ such that

$$\operatorname{ess\,inf}\{D(x) : x \in \mathbb{R}^n \setminus B_{R_1}(0)\} \ge \beta,$$

H3) (a) $A^{\frac{-2}{q(x)-2}}(x) \in L^1(\mathbb{R}^n)$,
(b) there is a positive constant s such that

$$\max\left\{\frac{2n}{n+2}, 1\right\} < s < \min\{p^-, q^-\} \quad \text{and} \quad D^{\frac{-s}{p(x)-s}}(x) \in L^1(B_{R_1}(0)).$$

The external term f is assumed to satisfy the conditions

H4) $$\begin{cases} f(x, t, s) : S_T \times \mathbb{R} \mapsto \mathbb{R} \text{ is globally Lipschitz-continuous with respect to } s \\ \text{and has linear growth: there is a constant } L \text{ such that} \\ \qquad |f(x, t, u) - f(x, t, v)| \le L|u - v| \quad \forall u, v \in \mathbb{R},\ (x, t) \in S_T, \\ \qquad |f(x, t, s)| \le L|s| + f_0(x, t) \quad \text{with some } f_0 \in L^2(S_T). \end{cases}$$

Assumptions H1)–H3) allow the coefficients D and A to vanish on a set of zero measure, moreover, it is necessary that $A(x) \to \infty$ as $|x| \to \infty$. An example of the equation whose coefficients meet all the above conditions is given by

$$u_t = \operatorname{div}\left(|x|^\alpha |\nabla u|^{p(x)-2}\nabla u\right) - |x|^\delta (1 + |x|^\gamma)|u|^{q(x)-2}u$$

with nonnegative constant exponents α, δ, γ satisfying the inequalities

$$0 \le \alpha < n\left(\frac{p^-}{s} - 1\right), \qquad 0 \le \delta < n\left(\frac{q^-}{2} - 1\right), \qquad \delta + \gamma > n\left(\frac{q^+}{2} - 1\right).$$