J. Math. Anal. Appl. 443 (2016) 295–312

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On convex intersection bodies and unique determination problems for convex bodies

Matthew Stephen

Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, Alberta, T6G 2G1, Canada

ARTICLE INFO

Article history:
Received 9 December 2015
Available online 17 May 2016
Submitted by J. Bastero

Keywords:
Intersection body
Cross-section body
Convex intersection body
Klee's problem

ABSTRACT

We describe a general result which ensures counter-examples for certain problems of unique determination for convex bodies. Using this result, we show a convex body $K \subset \mathbb{R}^n$ is not uniquely determined by its convex intersection body $CI(K)$, introduced by Meyer and Reisner in 2011.

1. Introduction

Let $K \subset \mathbb{R}^n$ be a *convex body*, a convex and compact subset which includes the origin as an interior point. Many classic problems in convex geometry are concerned with the unique determination, possibly up to congruency, of a convex body within some collection.

One well-known positive result is the Minkowski–Funk Section Theorem (see, for example, Corollary 3.9 in [9]), which may be stated in terms of intersection bodies. Recall that $L \subset \mathbb{R}^n$ is a *star body* if the closed line segment connecting the origin to every $x \in L$ is contained in L, and if its *radial function*

$$\rho_L(\xi) = \max\left\{a \geq 0 \middle| a\xi \in L\right\}, \qquad \xi \in S^{n-1},$$

is positive and continuous. Note that all convex bodies are also star bodies. We say L is *origin-symmetric* if $L = -L$. More generally, L is *centrally-symmetric* if one of its translates is origin-symmetric. The *intersection body* of L is the star body $IL \subset \mathbb{R}^n$ with radial function

$$\rho_{IL}(\xi) = \mathrm{vol}_{n-1}\left(L \cap \xi^{\perp}\right), \qquad \xi \in S^{n-1}.$$

E-mail address: mastephe@ualberta.ca.

http://dx.doi.org/10.1016/j.jmaa.2016.05.023
0022-247X/

Intersection bodies were first introduced by Lutwak in [10] in connection with the Busemann–Petty problem, and they continue to be an active area of research. Now, the aforementioned theorem is as follows:

Theorem 1 *(Minkowski–Funk Section Theorem). If K, $L \subset \mathbb{R}^n$ are origin-symmetric star bodies with $IK = IL$, then $K = L$.*

There are counter-examples for this result when K and L are not assumed to be origin-symmetric; see, for instance, [3]. It is natural to look for conditions for unique determination which hold in the absence of origin-symmetry.

Klee proposed such conditions in [8]. We will state Klee's problem in terms of cross-section bodies, which were introduced by Martini in [12]. Consider again a general convex body $K \subset \mathbb{R}^n$. The *cross-section body* of K is the star body $CK \subset \mathbb{R}^n$ with

$$\rho_{CK}(\xi) = \max_{t\in\mathbb{R}} \mathrm{vol}_{n-1}\left(K \cap \{\xi^\perp + t\xi\}\right), \qquad \xi \in S^{n-1}.$$

It is obvious that $IK \subset CK$, and by Brunn's Theorem, $IK = CK$ when K is origin-symmetric. In [11], it was proven that $IK = CK$ only if K is origin-symmetric. Now, Klee asked whether $CK = CL$ implies $K = L$, for (not necessarily origin-symmetric) convex bodies K, $L \subset \mathbb{R}^n$. This question was only recently proven in the negative in [4], where an explicit counter-example was constructed. Subsequently, [16] and [17] gave alternative constructions.

In general, intersection and cross-section bodies are not convex bodies. Busemann's Theorem implies that for a convex body K, IK is convex when K is origin-symmetric. If K is not origin-symmetric, then IK is not necessarily convex, nor is it necessary that CK is so when $n > 3$; see [2,13,1].

In [14], Meyer and Reisner introduced convex intersection bodies. Let $K \subset \mathbb{R}^n$ be a convex body. Let $g = g(K) \in K$ be the centroid of K, and let K^{*y} denote the *polar body* of K with respect to the point $y \in \mathrm{int}(K)$; that is,

$$K^{*y} = \left\{x \in \mathbb{R}^n \,\middle|\, \langle x - y, z - y\rangle \le 1 \ \forall\ z \in K\right\}.$$

The *convex intersection body* of K is the (a priori) star body $CI(K) \subset \mathbb{R}^n$ with

$$\rho_{CI(K)}(\xi) = \min\left\{\mathrm{vol}_{n-1}\left[\left(K^{*g}|\xi^\perp\right)^{*y}\right] \,\middle|\, y \in \mathrm{int}\left(K^{*g}|\xi^\perp\right)\right\}, \qquad \xi \in S^{n-1},$$

where $\cdot|\xi^\perp$ is the orthogonal projection onto the hyperplane perpendicular to ξ. The main result in [14] is that $CI(K)$ is always a convex body.

Furthermore, when the centroid of K is at the origin, the relationship between $CI(K)$ and IK parallels that between IK and CK. Indeed, with the assumption $g(K) = 0$, it was proved in [14] that $CI(K) \subset IK$ and $CI(K) = IK$ if and only if K is origin-symmetric.

Is a convex body uniquely determined, up to congruency, by its convex intersection body? If it is centrally-symmetric, then yes, as follows from above. Recall that a convex body K is infinitely *smooth* if its radial function is infinitely smooth on S^{n-1}. We will say K is a convex or star *body of rotation* if its radial function is *rotationally symmetric* about the x_1-axis; i.e.

$$\rho_K(\xi) = \rho_K(\eta) \quad \text{whenever} \quad \xi, \eta \in S^{n-1} \quad \text{and} \quad \langle \xi, e_1\rangle = \langle \eta, e_1\rangle,$$

where e_1 is the unit vector in the direction of the positive x_1-axis. We prove the following:

Theorem 2. *Let $n \in \mathbb{N}$, $n \ge 2$. There are infinitely smooth convex bodies of rotation K, $L \subset \mathbb{R}^n$ such that K is not centrally-symmetric, L is origin-symmetric, and $CI(K) = CI(L)$.*