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Estimates of the extremal solution for the bilaplacian with general nonlinearity

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ABSTRACT

Let $\lambda^* > 0$ denote the supremum possible value of λ such that $\{\Delta^2 u = \lambda f(u)$ in B_1, $u = \frac{\partial u}{\partial n} = 0$ on $\partial B_1\}$ has a classical solution, where B_1 is the unit ball in $\mathbb{R}^N$, n is the exterior unit normal vector, and $f \in C^1(\mathbb{R})$ is nondecreasing and satisfies $f(0) > 0$ and $f(t)/t \to +\infty$ as $t \to +\infty$. For $\lambda = \lambda^*$ this problem possesses a weak solution u^*, the so-called extremal solution. We establish the regularity of this extremal solution for $N \leq 10$. For $N \geq 11$ we establish that $\lim_{r\to 0} r^{\frac{N-8}{2}}(u^*)'(r) = \lim_{r\to 0} r^{\frac{N-10}{2}}u^*(r) = 0$ for $N \leq 19$ and $\lim_{r\to 0} r^{\frac{N-9}{2}}(u^*)'(r) = \lim_{r\to 0} r^{\frac{N-11}{2}}u^*(r) = 0$ for $N \geq 20$. Our regularity results do not depend on the specific nonlinearity f.

1. Introduction and main results

This paper is concerned with the stability of radially symmetric and decreasing solutions $u \in H^2(B_1)$ of

$$\begin{cases} \Delta^2 u = \lambda f(u) & \text{in } B_1, \\ u = \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial B_1, \end{cases} \tag{1.1_λ}$$

where B_1 is the unit ball of $\mathbb{R}^N$, n is the exterior unit normal vector, $\lambda \geq 0$ is a parameter, and $f \in C^1(\mathbb{R})$ satisfies

$$f \text{ is nondecreasing, } f(0) > 0 \text{ and } \lim_{t\to+\infty} \frac{f(t)}{t} = +\infty. \tag{1.2}$$

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By abuse of notation, we write $u(r)$ instead of $u(x)$, where $r = |x|$ and $x \in \mathbb{R}^N$. We denote by u' the radial derivative of a radial function u.

Definition. We say that $u \in L^1(B_1)$ is a weak solution of (1.1$_\lambda$) if $f(u) \in L^1(B_1, \delta(x)^2)$ and

$$\int\limits_{B_1} u\Delta^2\varphi = \lambda \int\limits_{B_1} f(u)\varphi,\ \forall \varphi \in C^4(\overline{B_1}),\ \varphi = \frac{\partial\varphi}{\partial n} = 0 \text{ on } \partial B_1, \tag{1.3}$$

where $\delta(x) = \operatorname{dist}(x, \partial B_1)$ denotes the distance to the boundary of B_1.

It is obvious that every C^4 classical solution of (1.1$_\lambda$) is a weak solution.

Definition. Let u be a solution of (1.1$_\lambda$), u is stable if

$$Q_u(\xi) := \int\limits_{B_1} \left\{ |\Delta\xi|^2 - \lambda f'(u)\xi^2 \right\} \geq 0,\ \forall \xi \in C_c^\infty(B_1). \tag{1.4}$$

Theorem 1.1 *([9]). There exists $\lambda^* < \infty$ such that:*

i) *If $\lambda \in [0, \lambda^*)$, (1.1$_\lambda$) admits a classical minimal solution u_λ.*
ii) *If $\lambda > \lambda^*$, there does not exist a classical solution.*
iii) *If $\lambda = \lambda^*$, there exists a weak solution $\lim\limits_{\lambda\to\lambda^*} u_\lambda = u^* \in L^1(B_1)$ of (1.1$_{\lambda^*}$), called the extremal solution.*

The minimal solutions of (1.1$_\lambda$) are radial and stable (see [9, Prop. 1]), and then u^* is also radial and stable.

In this paper, we study the regularity of the extremal solution u^*. For $f(u) = e^u$ Dávila et al. (see [4]) prove that u^* is bounded if $N \leq 12$, and u^* is unbounded if $N \geq 13$.

Theorem 1.2 *([9]). Assume that f satisfies (1.2). Let u^* be the extremal solution of (1.1$_\lambda$). If $N \leq 9$, then u^* is bounded.*

It leaves open the question of whether u^* is bounded for general nonlinearities f satisfying (1.2) in dimensions $10 \leq N \leq 12$.

In this paper we prove the boundedness of u^* in dimension $N = 10$. In addition, we also establish some estimates of the extremal solution near the origin for $N \geq 11$.

Theorem 1.3. *Assume that f satisfies (1.2). Let u^* be the extremal solution of (1.1$_\lambda$). We have that:*

i) *If $N = 10$, then u^* is bounded.*
ii) *If $11 \leq N \leq 19$, then*

$$\lim_{r\to 0} r^{\frac{N-8}{2}} (u^*)'(r) = \lim_{r\to 0} r^{\frac{N-10}{2}} u^*(r) = 0.$$

iii) *If $N \geq 20$, then*

$$\lim_{r\to 0} r^{\frac{N-9}{2}} (u^*)'(r) = \lim_{r\to 0} r^{\frac{N-11}{2}} u^*(r) = 0.$$