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On the existence of short trajectories of quadratic differentials related to generalized Jacobi polynomials with non-real varying parameters

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ABSTRACT

The motivation of this paper is to analyze to a large-degree the behavior of the Jacobi $\left(P_n^{\alpha_n,\beta_n}\right)$ polynomials when the parameters determining the polynomials are complex and depend on the degree n linearly. We show that the Cauchy transform of the limit (weak) of the root-counting measures of these polynomials satisfies quadratic algebraic equation. We investigate the existence of solutions to these equations as Cauchy transform of compactly supported positive measures. Any connected curve of the support of these measures (if exists) coincides with a horizontal trajectory of some quadratic differential. In this paper, we describe the trajectories of the family of quadratic differentials $\frac{\lambda^2(z-a)(z-b)}{(z^2-1)^2}dz^2$, and we give a necessary and sufficient condition on the complex numbers a, b, and λ for the existence of at least one finite critical trajectory.

1. Introduction

This paper is a completion of preceding works motivated by the large-degree analysis of the behavior of the Jacobi polynomials $P_n^{(\alpha,\beta)}$, when the parameters $\alpha, \beta \in \mathbb{C}$ depend on the degree n linearly. Recall that these polynomials can be given explicitly by (see [18])

$$P_n^{(\alpha,\beta)}(z) = 2^{-n} \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \binom{n+\beta}{k} (z-1)^k (z+1)^{n-k}, \tag{1}$$

where $\binom{\gamma}{k} = \frac{\gamma(\gamma-1)\ldots(\gamma-k+1)}{k!}$ for $(\gamma, k) \in \mathbb{C} \times \mathbb{N}$. Equivalently, the Jacobi polynomials can be defined by the well-known Rodrigues formula

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$$P_n^{(\alpha,\beta)}(z) = \frac{1}{2^n n!}(z-1)^{-\alpha}(z+1)^{-\beta}\left(\frac{d}{dz}\right)^n\left[(z-1)^{n+\alpha}(z+1)^{n+\beta}\right].$$

Moreover, they satisfy the differential equation

$$\left(1-z^2\right)y'' + (\beta-\alpha-(\alpha+\beta+2)z)\,y' + n\,(n+\alpha+\beta+1)\,y = 0, \tag{2}$$

and, the so-called three-terms recurrence relation

$$\begin{aligned} &2n\,(n+\alpha+\beta)\,(2n+\alpha+\beta-2)\,P_n^{(\alpha,\beta)}(z) = \\ &(2n+\alpha+\beta-1)\,[(2n+\alpha+\beta)\,(2n+\alpha+\beta-2)\,z+\alpha-\beta]\,P_{n-1}^{(\alpha,\beta)}(z) + \\ &-2\,(n+\alpha-1)\,(n+\beta-1)\,(2n+\alpha+\beta)\,P_{n-2}^{(\alpha,\beta)}(z)\,. \end{aligned} \tag{3}$$

Polynomials $P_n^{(\alpha,\beta)}$ are entire functions of the complex parameters α, β. The classical case is when $\alpha, \beta > -1$: in theses cases, the Jacobi polynomials are orthogonal on $[-1, 1]$ with respect to the weight function $(1-x)^\alpha(1+x)^\beta$, and then, their zeros are all simple, and belong to $(-1, 1)$. For general α, β, it was shown in [8] that there exists a contour (and then, by analyticity, many such contours) in the complex plane, in which we can associate a complex non-Hermitian orthogonality with $P_n^{(\alpha,\beta)}$. This non-Hermitian orthogonality is in fact the key feature in the study of the limit root location, the weak (via the Gonchar–Rakhmanov–Stahl (GRS) theory [5,16]) and the strong (via the Riemann–Hilbert (RH) steepest descent method of Deift–Zhou [3]) asymptotics of these polynomials on the whole plane.

We fix two complex numbers A and B, and consider the sequence

$$p_n(z) = P_n^{(\alpha_n,\beta_n)}(z)\,, \alpha_n = nA, \beta_n = nB. \tag{4}$$

The case $A, B \geq 0$ can be tackled using the standard tools related to varying orthogonality and equilibrium measures in an external field on $\mathbb{R}$, see e.g. [5], or [4]. The general situation $A, B \in \mathbb{R}$ was analyzed in [7–9]. The situation $A \notin \mathbb{R}$, $B > 0$ was analyzed in [10].

In this paper, we are interested in the situation when

$$A \notin \mathbb{R}, B \notin \mathbb{R}, A+B \notin \mathbb{R}. \tag{5}$$

We associate to each p_n its normalized root-counting measure

$$\mu_n = \frac{1}{n}\sum_{p_n(a)=0}\delta_a,$$

where δ_a is the Dirac measure supported at a, and the zeros are counted with their multiplicities. The limiting set of the zeros of the Jacobi polynomials p_n's is the unique short trajectory $\gamma_{A,B}$ of the quadratic differential

$$\varpi_{A,B} = -\frac{R_{A,B}(z)}{\left(z^2-1\right)^2}dz^2, \tag{6}$$

where

$$R_{A,B}(z) = (A+B+2)^2 z^2 + 2\left(A^2-B^2\right)z + (A-B)^2 - 4\,(A+B+1)\,. \tag{7}$$

The GRS method indicates that the sequence $\{\mu_n\}$ converges (as $n \longrightarrow \infty$) in the weak-* topology to the measure μ supported on $\gamma_{A,B}$ and given by