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Limit theorems for some Markov chains

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ABSTRACT

The exponential rate of convergence and the Central Limit Theorem for some Markov operators are established. These operators were efficiently used in some biological models (see Hille, Horbacz & Szarek (2015) [8]), which generalize the cell cycle model given by Lasota & Mackey (1999) [12].

1. Introduction

We are concerned with Markov operators corresponding to iterated function systems. The main goals of the paper are to prove exponential rate of convergence and establish the Central Limit Theorem (CLT). It should be indicated that the first result implies the second. The operators under consideration are more general than those used in [12]. The authors studied therein some cell cycle model, in which the rate of convergence is already evaluated by Wojewódka [22]. Hille, Horbacz & Szarek [8] proposed the generalization of the model and assured the existence of a unique invariant distribution in it. We have managed to evaluate the rate of convergence, which provides asymptotic stability at once, as well as allows to show the CLT. The

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results bring some information important from biological point of view. To get more details on biological background of the research, see [20] or [16].

In our paper we base on coupling methods introduced by Hairer [5]. In the same spirit, exponential rate of convergence was proven by Ślęczka [18] for classical iterated function systems (see also [6] or [10]). However, we use coupling methods not only to evaluate the rate of convergence. It turns out that properly constructed coupling measure, if combined with the results for stationary ergodic Markov chains given by Maxwell & Woodroofe [14], is crucial in the proof of the CLT, too. If we have the coupling measure already constructed, the proof of the CLT is brief and less technical than typical proofs based on Gordin's martingale approximation, e.g. the one proposed by Komorowski & Walczuk [11]. It is worth mentioning here that coupling methods are adapted to an auxiliary model, described by some non-homogeneous Markov chain. While reading the paper, one may see that it is a bright idea to express the Markov operator of interest by means of an auxiliary one.

Similar approach can also help to establish the Law of the Iterated Logarithm (LIL). The proof of the LIL is provided in [9]. Some ideas useful for proving it are adapted from [2]. However, using an appropriate coupling measure, again, makes the proof much easier.

The organization of the paper goes as follows. Section 2 introduces basic notation and definitions that are needed throughout the paper. Most of them are adapted from [1,15,13,19]. Mathematical derivation of the generalized cell cycle model is provided in Section 3. The main theorems (Theorem 1 and Theorem 2) are also formulated there. Sections 5–7 are devoted to the construction of coupling measure for iterated function systems. Thanks to the results presented in Section 8 we are finally able to present the proofs of main theorems. Indeed, the exponential rate of convergence is established in Section 9 and the CLT – in Section 10.

2. Notation and basic definitions

Let (X, ϱ) be a Polish space. We denote by B_X the family of all Borel subsets of X. Let $B(X)$ be the space of all bounded and measurable functions $f : X \to R$ with the supremum norm and write $C(X)$ for its subspace of all bounded and continuous functions with the supremum norm.

We denote by $M(X)$ the family of all Borel measures on X and by $M_{\text{fin}}(X)$ and $M_1(X)$ its subfamilies such that $\mu(X) < \infty$ and $\mu(X) = 1$, respectively. Elements of $M_{\text{fin}}(X)$ which satisfy $\mu(X) \leq 1$ are called sub-probability measures. To simplify notation, we write

$$\langle f, \mu \rangle = \int_X f(x)\mu(dx) \quad \text{for } f : X \to R,\ \mu \in M(X).$$

An operator $P : M_{\text{fin}}(X) \to M_{\text{fin}}(X)$ is called a Markov operator if

(i) $P(\lambda_1\mu_1 + \lambda_2\mu_2) = \lambda_1 P\mu_1 + \lambda_2 P\mu_2 \quad$ for $\lambda_1, \lambda_2 \geq 0,\ \mu_1, \mu_2 \in M_{\text{fin}}(X)$;
(ii) $P\mu(X) = \mu(X) \quad$ for $\mu \in M_{\text{fin}}(X)$.

Markov operator P for which there exists a linear operator $U : B(X) \to B(X)$ such that

$$\langle Uf, \mu \rangle = \langle f, P\mu \rangle \quad \text{for } f \in B(X),\ \mu \in M_{\text{fin}}(X)$$

is called a regular operator. We say that a regular Markov operator is Feller if $U(C(X)) \subset C(X)$. Every Markov operator P may be extended to the space of signed measures on X denoted by $M_{\text{sig}}(X) = \{\mu_1 - \mu_2 :\ \mu_1, \mu_2 \in M_{\text{fin}}(X)\}$. For $\mu \in M_{\text{sig}}(X)$, we denote by $\|\mu\|$ the total variation norm of μ, i.e.,

$$\|\mu\| = \mu^+(X) + \mu^-(X),$$