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Measure of complete dependence of random vectors ☆

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ABSTRACT

In this work, we define a family of measures of complete dependence of absolutely continuous random vectors extended those of random variables. We show that these measures satisfy a suitable set of properties to be called measures of complete dependence. Computational examples are also given.

1. Introduction

One method to qualify dependence between random elements is to rank them according to how close one is a function of another. Several measures based on this idea have been proposed. Notable works are those of Trutschnig [6], Dette et al. [1], and Siburg and Stoimenov [3]. All of these three works define measures of dependence between continuous random variables. The latter two define measures as functions of copulas associated with those random variables. Trutschnig [6], on the other hand, defines the measure in term of the Markov operators associated with those random variables. Nevertheless, Trutschnig's measure can be written in term of copulas also.

A measure of complete dependence of random vectors has also been recently proposed by Tasena and Dhompongsa [5]. They showed that the constructed measure satisfies several desired properties of measure of complete dependence. Nevertheless, the constructed measure has at least two flaws. First, the constructed measure can not be viewed as a distance between copula and the copula associated with independent random vectors. Therefore, the interpretation of the measure is unclear. Second, the constructed measure depends on the marginal distributions. This means the measure might not be suitable to compare dependence between random vectors with different distributions (see the discussion section in [5]).

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In this work, we define a measure that will overcome these drawbacks. We also show that the constructed measure satisfies all applicable properties considered in [5] as well as other properties.

This work is organized as follows. In section 2, we discuss terminologies and notations used throughout this work as well as previously defined measures. In section 3, we provide the definition of our measure and the proof that it satisfies several desirable properties. Example calculations are also provided at the end of the section.

2. Preliminaries

Let $(\Omega, A, \mathbb{P})$ be a probability space. A *random variable* is a measurable function from Ω to $\mathbb{R}$. An *n-dimensional random vector* is a measurable function from Ω to $\mathbb{R}^n$. Coordinate-wise a random vector can be viewed as a tuple of random variables.

Each random vector X is associated with a *distribution function* F_X defined by $F_X(x) := \mathbb{P}(X \leq x)$ for all x. By identifying the space $\mathbb{R}^{n_1} \times ... \times \mathbb{R}^{n_k}$ with $\mathbb{R}^{n_1+\cdots+n_k}$, we may view a tuple $(X_1, ..., X_k)$ of random vectors as a random vector. In this case, the distribution function of $(X_1, ..., X_k)$ will be called the *joint distribution function* of $X_1, ..., X_k$ and will be denoted by $F_{X_1,...,X_k}$ instead of $F_{(X_1,...,X_k)}$ to emphasize the fact that we consider $X_1, ..., X_k$ separately. Moreover, we call the distribution functions of $X_1, ..., X_k$ the *marginals* of $F_{X_1,...,X_k}$.

For any random vectors X and Y, the *conditional distribution* $F_{Y|X}$ of Y given X is defined by

$$F_{Y|X}(y|x) := \lim_{h \searrow 0} \frac{\mathbb{P}(x - h < X \leq x + h, Y \leq y)}{\mathbb{P}(x - h < X \leq x + h)}$$

for all x and y. It can be proved that $\int F_{Y|X}(y|x)\mathrm{d}F_X(x) = F_Y(y)$ for all y. It is well-known that random vectors X and Y are independent if and only if $F_{X,Y}(x, y) = F_X(x)F_Y(y)$ for all x and y if and only if $F_{Y|X}(y|x) = F_Y(y)$ for all x and y.

A random vector X is said to be *(absolutely) continuous* if its distribution function has a *density*, that is, a function f_X in which

$$F_X(x) = \int_{(-\infty,x]} f_X \mathrm{d}\lambda$$

for all x. Here λ denotes the *Lebesgue measure* on the real line. **In this work, we will only consider random vectors with absolutely continuous distribution functions**.

A random vector U is said to be *uniform* if its distribution function F_U is the product function Π, that is,

$$F_U(u) = \Pi(u) := \Pi_{i=1}^n u_i$$

for all $u = (u_1, ..., u_n) \in [0, 1]^n$.

A joint distribution function of uniform random variables is called a *copula* and a joint distribution function of uniform random vectors is called a *linkage*. Note that a linkage is always a copula.

For each distribution function F_X of a random vector $X = (X_1, ..., X_n)$, define a *transformation* Ψ_{F_X} : $\mathbb{R}^n \to [0, 1]^n$ by letting

$$\Psi_{F_X}(x_1, ..., x_n) = \big(F_{X_1}(x_1), F_{X_2|X_1}(x_2|x_1), ..., F_{X_n|(X_1,...,X_{n-1})}(x_n|(x_1, ..., x_{n-1}))\big)$$

for all $(x_1, ..., x_n) \in \mathbb{R}^n$. It is known that $U = \Psi_{F_X}(X)$ has uniform distribution. Next, let $X_1, ..., X_k$ be random vectors of dimensions $n_1, ..., n_k$, respectively, and $\Psi_i = \Psi_{F_{X_i}}$ be the distribution transformation