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Complex symmetric composition operators on H^2

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ABSTRACT

In this paper, we find complex symmetric composition operators on the classical Hardy space H^2 whose symbols are linear-fractional but not automorphic. In doing so, we answer a recent question of Noor, and partially answer the original problem posed by Garcia and Hammond.

1. Introduction

In this paper, we are interested in composition operators on the classical *Hardy space* H^2. This Hilbert space consists of the set of analytic functions $f(z) = \sum_{n=0}^{\infty} a_n z^n$ on the open unit disk $\mathbb{D}$ such that

$$\|f\|^2 = \sum_{n=0}^{\infty} |a_n|^2 < \infty.$$

A *composition operator* C_φ on H^2 is given by $C_\varphi f = f \circ \varphi$. When φ is an analytic self-map of $\mathbb{D}$, the operator C_φ is necessarily bounded. A *Toeplitz operator* T_ψ on H^2 is given by $T_\psi f = \psi f$ when $\psi \in H^\infty$, the space of bounded analytic functions on $\mathbb{D}$. We occasionally write $W_{\psi,\varphi} = T_\psi C_\varphi$ and call such an operator a *weighted composition operator*. When φ is linear-fractional, C_φ^* has a simple form which involves composition operators and (adjoints of) Toeplitz operators, which we will put to use later in this paper.

The weak normality properties of composition operators and weighted composition operators on H^2 have been of great interest. For example, work has been done on when C_φ and $W_{\psi,\varphi}$ are normal [1], subnormal [4], and cohyponormal [6]. In [7], Garcia and Hammond posed the following problem: "Characterize all complex symmetric composition operators C_φ on the classical Hardy space H^2." An operator T on a Hilbert space $\mathcal{H}$

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is complex symmetric if there exists a conjugate-linear, isometric involution J on $\mathcal{H}$ so that $T = JT^*J$. We call such an operator J a *conjugation*. For applications and further information about complex symmetric operators, see [8,9] and [10].

One of the first known classes of complex symmetric operators was that of the normal operators [8]. Hence, any normal composition operator induced by $\varphi(z) = az$ with $|a| \leq 1$ [5, Theorem 8.2] is complex symmetric. Garcia and Hammond showed that [7, Proposition 2.5] any complex symmetric composition operator induced by a non-constant φ must be univalent.

Due to a result of Garcia and Wogen [10], any automorphism of order 2 is complex symmetric. In [2], Bourdon and Noor found that other automorphisms cannot be complex symmetric except possibly for the unsolved case when φ is an elliptic automorphism of order 3. In [12], Noor found the exact conjugation needed for the involutive disk automorphisms to induce complex symmetric composition operators. Jung, Kim, Ko, and Lee were thought to have found a non-automorphic example in [11], but were disproved by Noor in [13]. Noor stated that even a narrower question remains open: does there exist a non-constant and non-automorphic symbol φ for which C_φ is complex symmetric but not normal on H^2?

In this paper, we give examples of linear-fractional, non-automorphic maps φ that induce complex symmetric composition operators on H^2. In doing so, we answer positively the question of Noor, and we answer in part the question originally posed by Garcia and Hammond. We now state our main theorem.

Main Theorem. *Let $\sigma(z) = az + c$ and $\varphi(z) = az/(1 - cz)$ be analytic self-maps of $\mathbb{D}$ such that neither of them is the identity map, $\operatorname{id}(z) = z$. The operator C_σ, respectively C_φ, is complex symmetric on H^2 if and only if σ, respectively φ, has Denjoy–Wolff point in $\mathbb{D}$ and no fixed point on the boundary.*

Throughout Section 2, $b = c/(1 - a)$ will appear often because it is the Denjoy–Wolff point of σ. Our strategy will be to first prove that C_σ is complex symmetric with real fixed point b, and from there, we will extend to complex fixed points and then consider C_φ. To that end, here we mention the Cowen adjoint formula for composition operators with linear-fractional symbol [5, Theorem 9.2]. The theorem is well-known but we restate it here due to its frequent use throughout the paper.

Theorem 1.1 *(Cowen adjoint formula). If $\sigma = \frac{az+b}{cz+d}$ is an analytic self-map of $\mathbb{D}$, then on H^2, $C_\sigma^* = T_g C_\varphi T_h^*$, where $g = 1/(-\overline{b}z + \overline{d})$, $h = cz + d$, and $\varphi = (\overline{a}z - \overline{c})/(-\overline{b}z + \overline{d})$, and g, h necessarily belong to H^∞. Moreover, φ is an analytic self-map of $\mathbb{D}$.*

The function φ is called the *Krein adjoint* of σ. It follows from the proof of Theorem 1.1, if s is a fixed point of σ, then $1/\overline{s}$ is a fixed point of φ.

2. Results

Let J be the operator defined by $(Jf)(z) = \overline{f(\overline{z})}$. Then J is a conjugation. We will consider when J commutes with other involutions, so that the involution in our construction is conjugate-linear.

Proposition 2.1. *Suppose J_1 and J_2 are isometric involutions which commute, and one of them is linear and the other is conjugate-linear. Then $J_1 J_2$ is a conjugation.*

Proof. A product of isometries is an isometry. Since $J_1J_2J_1J_2 = J_1J_2J_2J_1 = J_1J_1 = I$, J_1J_2 is an involution. Without loss of generality, say J_1 is conjugate-linear. Then $J_1J_2(af + bg) = J_1(aJ_2f + bJ_2g) = \overline{a}J_1J_2f + \overline{b}J_1J_2g$. Therefore, J_1J_2 is a conjugation. $\square$

Proposition 2.2. *Suppose that $\psi \in H^\infty$ and $\varphi : \mathbb{D} \to \mathbb{D}$ both map $(-1, 1)$ into itself. Then J commutes with T_ψ and C_φ, and therefore also with $W_{\psi,\varphi}$.*