J. Math. Anal. Appl. 442 (2016) 31–49

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

On the X-ray transform of planar symmetric 2-tensors

Kamran Sadiq [a,*], Otmar Scherzer [b,a], Alexandru Tamasan [c]

[a] Johann Radon Institute of Computational and Applied Mathematics (RICAM), Altenbergerstrasse 69, 4040 Linz, Austria
[b] Computational Science Center, Oskar-Morgenstern-Platz 1, 1090 Vienna, Austria
[c] Department of Mathematics, University of Central Florida, Orlando, 32816 FL, USA

ARTICLE INFO

Article history:
Received 20 March 2015
Available online 25 April 2016
Submitted by H. Kang

Keywords:
X-ray transform of symmetric tensors
Attenuated X-ray transform
A-analytic maps
Hilbert transform
Boundary rigidity problem

ABSTRACT

In this paper we study the attenuated X-ray transform of 2-tensors supported in convex bounded subsets with sufficiently smooth boundary in the Euclidean plane. We characterize its range and reconstruct all possible 2-tensors yielding identical X-ray data. The characterization is in terms of a Hilbert-transform associated with A-analytic maps in the sense of Bukhgeim.

1. Introduction

This paper concerns the range characterization of the attenuated X-ray transform of symmetric 2-tensors in the plane. Range characterization of the non-attenuated X-ray transform of functions (0-tensors) in the Euclidean space has been long known [8,9,14], whereas in the case of a constant attenuation some range conditions can be inferred from [13,1,2]. For a varying attenuation the two dimensional case has been particularly interesting with inversion formulas requiring new analytical tools: the theory of A-analytic maps originally employed in [3], and ideas from inverse scattering in [17], see also [16]. Constraints on the range for the two dimensional X-ray transform of functions were given in [18,4], and a range characterization based on Bukhgeim's theory of A-analytic maps was given in [23].

Inversion of the X-ray transform of higher order tensors has been formulated directly in the setting of Riemmanian manifolds with boundary [26]. The case of 2-tensors appears in the linearization of the boundary rigidity problem. It is easy to see that injectivity can hold only in some restricted class: e.g., the class of solenoidal tensors. For two dimensional simple manifolds with boundary, injectivity with in the

* Corresponding author.
E-mail addresses: kamran.sadiq@oeaw.ac.at (K. Sadiq), otmar.scherzer@univie.ac.at (O. Scherzer), tamasan@math.ucf.edu (A. Tamasan).

http://dx.doi.org/10.1016/j.jmaa.2016.04.018
0022-247X/

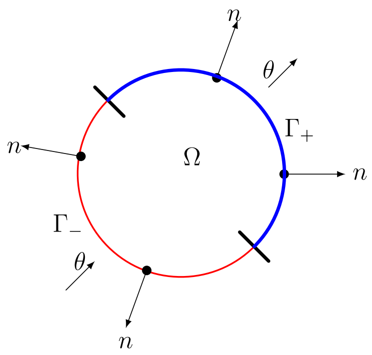

Fig. 1. Definition of $\Gamma_\pm$.

solenoidal tensor fields has been established fairly recent: in the non-attenuated case for 0- and 1-tensors we mention the breakthrough result in [22], and in the attenuated case in [25]; see also [10] for a more general weighted transform. Inversion for the attenuated X-ray transform for solenoidal tensors of rank two and higher can be found in [19], with a range characterization in [20]. In the Euclidean case we mention an earlier inversion of the attenuated X-ray transform of solenoidal tensors in [11,12]; however this work does not address range characterization.

Different from the recent characterization in terms of the scattering relation in [20], in this paper the range conditions are in terms of the Hilbert-transform for A-analytic maps introduced in [23,24], built on work in [27–29]. Our characterization can be understood as an explicit description of the scattering relation in [21,19,20] particularized to the Euclidean setting. In the sufficiency part we reconstruct all possible 2-tensors yielding identical X-ray data; see (30) for the non-attenuated case and (82) for the attenuated case.

For a real symmetric 2-tensor $\mathbf{F} \in L^1(\mathbb{R}^2; \mathbb{R}^{2\times 2})$,

$$\mathbf{F}(x) = \begin{pmatrix} f_{11}(x) & f_{12}(x) \\ f_{12}(x) & f_{22}(x) \end{pmatrix}, \quad x \in \mathbb{R}^2, \tag{1}$$

and a real valued function $a \in L^1(\mathbb{R}^2)$, the *a-attenuated X-ray transform of* $\mathbf{F}$ is defined by

$$X_a\mathbf{F}(x,\theta) := \int_{-\infty}^{\infty} \langle \mathbf{F}(x+t\theta)\,\theta, \theta\rangle \exp\left\{ -\int_t^{\infty} a(x+s\theta)ds \right\} dt, \tag{2}$$

where θ is a direction in the unit sphere $\mathbf{S}^1$, and $\langle\cdot,\cdot\rangle$ is the scalar product in $\mathbb{R}^2$. For the non-attenuated case $a \equiv 0$ we use the notation $X\mathbf{F}$.

In this paper, we consider $\mathbf{F}$ to be defined on a strongly convex bounded set $\Omega \subset \mathbb{R}^2$ with vanishing trace at the boundary Γ; further regularity and the order of vanishing will be specified in the theorems. In particular, in the attenuated case we assume that Γ is $C^{2,\alpha}$, $\alpha > \frac{1}{2}$ smooth. We also assume $a > 0$ in $\overline{\Omega}$.

For any $(x,\theta) \in \overline{\Omega} \times \mathbf{S}^1$ let $\tau(x,\theta)$ be length of the chord in the direction of θ passing through x. Let also consider the incoming ($-$), respectively outgoing ($+$) submanifolds of the unit bundle restricted to the boundary (see Fig. 1)

$$\Gamma_\pm := \{(x,\theta) \in \Gamma \times \mathbf{S}^1 : \pm\theta\cdot n(x) > 0\}, \tag{3}$$

and the variety

$$\Gamma_0 := \{(x,\theta) \in \Gamma \times \mathbf{S}^1 : \theta\cdot n(x) = 0\}, \tag{4}$$

where $n(x)$ denotes outer normal.