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The multiplicity of positive solutions for a class of nonlocal elliptic problem ☆

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A B S T R A C T

In this paper, using the theory of fixed point index, we prove some results on the multiplicity of positive solutions for a class of nonlocal elliptic problem.

1. Introduction

In this paper, we consider the following nonlocal elliptic problem

$$\begin{cases} -a\left(\int\limits_{\Omega} |u(x)|^{\gamma} dx\right) \Delta u = \lambda f(x,u), \quad x \text{ in } \Omega, \\ u(x) > 0, \quad x \text{ in } \Omega, \\ u(x) = 0, \quad x \text{ on } \partial\Omega, \end{cases} \tag{1.1$_\lambda$}$$

where $\Omega \subseteq R^N$ $(N \geq 1)$ is a smooth and bounded domain, $\gamma \in (0, +\infty)$, $a : [0, +\infty) \to (0, +\infty)$ and $f : \overline{\Omega} \times (0, +\infty)$ are given functions whose properties will be introduced later.

In [7], Chipot and Lovat considered the following model problem

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$$\begin{cases} u_t - a\left(\int\limits_\Omega u(z,t)dz\right)\Delta u = f, \quad \text{in } \Omega \times (0,T), \\ u(x,t) = 0, \quad \text{on } \Gamma \times (0,T), \\ u(x,0) = u_0(x), \quad \text{on } \Omega. \end{cases} \tag{1.2}$$

Here Ω is a bounded open subset in R^N, $N \geq 1$ with smooth boundary Γ, T is some arbitrary time. $(1.1)_\lambda$ is the elliptic version to (1.2). The diffusion coefficient a is some function from R into $(0, +\infty)$ which depends on the entire population in the domain rather than on the local density and u could describe the density of a population subject to spreading. And if $\gamma = 2$, we get the well-known Carrier's Equation. The problems like $(1.1)_\lambda$, which appear in some applied mathematics, attract a lot of attention and there are many results on the existence of positive solutions. For instance, by establishing comparison principles, using the results on fixed point index theory, sub-supersolution methods, Corrêa obtained the existence of at least one positive solution for $(1.1)_\lambda$ in [9] when $f(x,u) = H(x)f(u)$ and for the special case $\gamma = 1$, some results on the existence of at least one weak positive solution to $(1.1)_\lambda$ were presented also in [11]; in [13], Corrêa et al. considered the special case

$$\begin{cases} -\Delta u = u^q f(\lambda, \int_\Omega u^r), \quad x \text{ in } \Omega, \\ u > 0, \quad x \text{ in } \Omega, \\ u = 0, \quad x \text{ on } \partial\Omega, \end{cases} \tag{1.3}$$

and by transforming (1.3) into an algebraic nonlinear equation, gave a complete description of the set of positive solutions. Very recently, an interesting result on the following problems was obtained

$$\begin{cases} -a\left(\int\limits_\Omega |u(x)|^\gamma dx\right)\Delta u = h_1(x,u) f\left(\int\limits_\Omega |u(x)|^p dx\right) + h_2(x,u) g\left(\int\limits_\Omega |u(x)|^r dx\right), \quad x \text{ in } \Omega, \\ u = 0, \quad x \text{ on } \partial\Omega, \end{cases} \tag{1.4}$$

where $\gamma, r, p \geq 1$ and in which Alves et al. showed the existence of solution for some classes of nonlocal problems without of the monotonicity of f (see [3]).

The diffusion coefficient a can be generalized as $A(x; u) : \Omega \times L^p(\Omega) \to R$. In [6], Chipot and Corrêa considered the functional elliptic problems

$$\begin{cases} -A(x;u)\Delta u = \lambda f(u), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.5}$$

and by the Schauder fixed point theorem, they got the existence of at least one positive solution to (1.5). And moreover, by using a comparison principles, the existence of n distinct solutions was presented if f has n loops, see [8]. A more generalized equation is as follows

$$\begin{cases} -\mathrm{div}(A(x;u)\nabla u) = \lambda f(u), \quad x \in \Omega, \\ u|_{\partial\Omega} = 0, \end{cases} \tag{1.6}$$

and by methods of approximation and Schauder fixed point theorem, Roy obtained existence of nontrivial positive solutions to (1.6) under the lack of maximum principle, see [23]. We can notice that the following condition is necessary in [6,8,23]

$$0 < a_0 \leq A(x;u) \leq a_\infty < +\infty.$$