J. Math. Anal. Appl. 442 (2016) 189–205

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

A Gamma convergence approach to the critical Sobolev embedding in variable exponent spaces

Julián Fernández Bonder [a], Nicolas Saintier [b,*], Analia Silva [c]

[a] IMAS – CONICET and Departamento de Matemática, FCEyN – Universidad de Buenos Aires, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina

[b] CONICET and Dpto Matemática, FCEyN – Univ. de Buenos Aires, Ciudad Universitaria, Pabellón I (1428) Buenos Aires, Argentina

[c] IMASL – CONICET and Departamento de Matemática, Universidad Nacional de San Luis (5700) San Luis, Argentina

A R T I C L E I N F O

Article history:
Received 20 November 2015
Available online 26 April 2016
Submitted by V. Radulescu

Keywords:
Sobolev embedding
Variable exponents
Critical exponents
Concentration compactness

A B S T R A C T

In this paper, we study the critical Sobolev embeddings $W^{1,p(\cdot)}(\Omega) \subset L^{p^*(\cdot)}(\Omega)$ for variable exponent Sobolev spaces from the point of view of the Γ-convergence. More precisely we determine the Γ-limit of subcritical approximation of the best constant associated with this embedding. As an application we provide a sufficient condition for the existence of extremals for the best constant.

1. Introduction

The purpose of this paper is to analyze the Sobolev immersion theorem for variable exponent spaces in the critical range from the point of view of the Γ-convergence. Our motivation comes from the existence problem for extremals of these immersions. By extremals we mean functions $u \in W_0^{1,p(\cdot)}(\Omega)$ where the infimum

$$S = S(p(\cdot), q(\cdot), \Omega) := \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)}} \tag{1.1}$$

is attained. Here Ω is a smooth bounded subset of $\mathbb{R}^n$, and $p, q : \overline{\Omega} \to \mathbb{R}$ are two functions satisfying the following assumptions:

* Corresponding author.
E-mail addresses: jfbonder@dm.uba.ar (J. Fernández Bonder), nsaintie@dm.uba.ar (N. Saintier), acsilva@unsl.edu.ar (A. Silva).
URLs: http://mate.dm.uba.ar/~jfbonder (J. Fernández Bonder), http://mate.dm.uba.ar/~nsaintie (N. Saintier).

http://dx.doi.org/10.1016/j.jmaa.2016.04.021
0022-247X/

(H1) p is Log-Hölder continuous on Ω (see (2.2) below), $q \in C(\overline{\Omega})$,
(H2) $1 < p_- := \inf_{\overline{\Omega}} p \le p_+ := \sup_{\overline{\Omega}} p < n$,
(H3) $1 \le q(x) \le p^*(x) := np(x)/(n - p(x))$ for any $x \in \overline{\Omega}$.

We refer to the next section for the definition and basic properties of the variable exponent Sobolev spaces appearing in (1.1). Notice that the exponent p^* is critical from the Sobolev point of view. We shall also assume that

(H4) the set $\mathcal{A} := \{x \in \overline{\Omega}: q(x) = p^*(x)\}$ is non-empty.

Because of (H4) the embedding of $W^{1,p(\cdot)}(\Omega)$ into $L^{q(\cdot)}(\Omega)$ is not compact, making non-trivial the problem of existence of an extremal for S.

This problem was recently treated in [14] where the authors provide sufficient conditions to ensure the existence of such extremals. The approach in [14] was the so-called *direct method of the calculus of variations*. That is, they considered a minimizing sequence for S and find a sufficient condition that ensured the compactness of such sequence.

In this paper, we follow a different approach. Instead of looking for minimizing sequences for S, we approximate the critical problems by subcritical ones, where the existence of extremals is easily obtained, and then pass to the limit. In fact, following G. Palatucci in [29] and [30] where the constant exponent case is studied, we want to determine the asymptotic behavior in the sense of the Γ-convergence of the subcritical approximations

$$S_\varepsilon := S(p(\cdot), q(\cdot) - \varepsilon, \Omega) = \inf_{v \in W_0^{1,p(\cdot)}(\Omega)} \frac{\|\nabla v\|_{p(\cdot)}}{\|v\|_{q(\cdot)-\varepsilon}}, \quad \varepsilon > 0,$$

and then deduce the behavior of their associated extremals u_ε. We thus introduce the functional $F_\varepsilon : \mathcal{B}(\Omega) \to \mathbb{R}$, $\varepsilon > 0$, defined by

$$F_\varepsilon(u) := \int_\Omega |u|^{q(\cdot)-\varepsilon}\, dx,$$

where

$$\mathcal{B}(\Omega) := \left\{ u \in W_0^{1,p(\cdot)}(\Omega),\ \|\nabla u\|_{p(\cdot),\Omega} \le 1 \right\}, \tag{1.2}$$

with the purpose of finding its Γ-limit as $\varepsilon \to 0$.

This approach not only provides us with the existence of extremals for the critical embeddings but also gives us the asymptotic behavior of the subcritical extremals as the exponent q reaches a critical one. As in the constant exponent case, a concentration phenomenon occurs in the sense that the subcritical extremals concentrate at some point. In the constant exponent case the location of this point is related to the geometry of Ω via its Robin function (see e.g. [18]). The Gamma convergence turns out to be a useful tool in such analysis as was shown in [2] and in general in the study of the asymptotic behavior of variational problem (see e.g. [7]). On the other hand the study of such concentration phenomena in the variable exponent setting is a recent and rapidly growing area (see e.g. [1,13,14,16,15,19,24]). In particular the results in [13,16,15] let us think that the location of the concentration point may result of interplay between the exponents p and q on the one hand, and on the geometry of Ω on the other hand. The results of this paper are a first step toward a finer comprehension of the concentration phenomenon in the variable exponent setting.

In view of the concentration–compactness principle stated in (2.7)–(2.9) below, it turns out to be convenient to extend F_ε to the space