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The Bishop–Phelps–Bollobás property for operators from c_0 into some Banach spaces [☆]

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This paper is dedicated to our dear colleague Richard Aron

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ABSTRACT

We exhibit a new class of Banach spaces Y such that the pair (c_0, Y) has the Bishop–Phelps–Bollobás property for operators. This class contains uniformly convex Banach spaces and spaces with the property β of Lindenstrauss. We also provide new examples of spaces in this class.

1. Introduction

The well-known Bishop–Phelps Theorem [8] states that the set of norm attaining (continuous and linear) functionals on a Banach space is dense in its topological dual. After this result was proved, a lot of attention was devoted to extend it to operators (see [1,11,17], for instance).

In 1970, Bollobás showed the following "quantitative version" which is now called Bishop–Phelps–Bollobás Theorem [9]. As usual, for a normed space X, we denote by B_X and S_X the closed unit ball and the unit sphere of X, respectively. By X^* we stand the topological dual of X.

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Theorem 1.1 *(Bishop–Phelps–Bollobás Theorem). (See [10, Theorem 16.1].) Let X be a Banach space and $0 < \varepsilon < 1$. Given $x \in B_X$ and $x^* \in S_{X^*}$ with $|1 - x^*(x)| < \frac{\varepsilon^2}{4}$, there are elements $y \in S_X$ and $y^* \in S_{X^*}$ such that $y^*(y) = 1$, $\|y - x\| < \varepsilon$ and $\|y^* - x^*\| < \varepsilon$.*

For a refinement of the above result see [13, Corollary 2.4(a)]. In 2008 Acosta, Aron, García and Maestre initiated the study of parallel versions of this result for operators [3]. For two normed spaces X and Y over the scalar field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$), $\mathcal{L}(X, Y)$ denotes the space of (bounded and linear) operators from X into Y, endowed with the usual operator norm. For an operator $T \in \mathcal{L}(X, Y)$, T^t denotes the transpose of T.

Definition 1.2. (See [3].) Let X and Y be both either real or complex Banach spaces. It is said that the pair (X, Y) *has the Bishop–Phelps–Bollobás property for operators* (*BPBp*), if for any $\varepsilon > 0$ there exists $\eta(\varepsilon) > 0$ such that for any $T \in S_{\mathcal{L}(X,Y)}$, if $x \in S_X$ is such that $\|Tx\| > 1 - \eta(\varepsilon)$, then there exist an element u in S_X and an operator S in $S_{\mathcal{L}(X,Y)}$ satisfying the following conditions

$$\|Su\| = 1, \quad \|u - x\| < \varepsilon \quad \text{and } \|S - T\| < \varepsilon.$$

It was shown that the pair (X, Y) has the *BPBp* whenever X and Y are finite dimensional spaces [3, Proposition 2.4]. It was also proved that (X, Y) has the *BPBp* for every Banach space X, whenever Y has the property β of Lindenstrauss [3, Theorem 2.2]. A characterization of the Banach spaces Y such that the pair (ℓ_1, Y) has the *BPBp* was also provided [3, Theorem 4.1].

However, up to now there is no characterization of the spaces Y such that (c_0, Y) has the *BPBp*. First let us notice that the previous property is not trivially satisfied for every Banach space Y (see [7, Example 4.1]). Now we are going to mention some results known in the real case. Acosta et al. showed that (ℓ_∞^n, Y) has the *BPBp* for every nonnegative integer n whenever Y is uniformly convex [3]. Kim proved that the pair (c_0, Y) has the *BPBp* if Y is a uniformly convex Banach space [15]. There is a characterization of the Banach spaces Y such that (ℓ_∞^3, Y) has the *BPBp* for operators [5]. Moreover, Kim, Lee and Lin proved that the pair (L_∞, Y) has the *BPBp* for operators, whenever Y is uniformly convex [16].

Recently, Acosta showed that for the complex case the pair $(\mathcal{C}_0(L), Y)$ has the *BPBp* for every complex uniformly convex space Y and any locally compact Hausdorff topological space L [2].

On the other hand, for real or complex spaces, Aron, Cascales and Kozhushkina showed that the pair $(X, \mathcal{C}_0(L))$ has the *BPBp* for any locally compact Hausdorff space L in case that X is Asplund [6]. Later, Cascales, Guirao and Kadets extended this result to uniform algebras [12]. From here it follows that $(c_0, \mathcal{C}_0(L))$ has the *BPBp*.

In the real case, it was shown that the pair $(\mathcal{C}_0(L), \mathcal{C}_0(S))$ has the *BPBp* for any locally compact Hausdorff spaces L and S [4]. It is not known whether or not the parallel result holds in the complex case.

In this paper, we provide a new class of Banach spaces Y, containing uniformly convex spaces and spaces with the property β of Lindenstrauss, and such that the pair (c_0, Y) satisfies the Bishop–Phelps–Bollobás property for operators. Hence, spaces in this class can be very different from $\mathcal{C}_0(L)$. Moreover, elements in that class are not necessarily neither uniformly convex spaces nor spaces with the property β of Lindenstrauss.

2. The main result

The Banach spaces Y for which (ℓ_1, Y) has the *BPBp* for operators have been characterized in [3]. However nowadays it is considered as a main question in this subject to characterize the Banach spaces Y such that (c_0, Y) has the *BPBp* for operators.

As we already mentioned, our goal is to provide a new class of Banach spaces Y such that the pair (c_0, Y) has the Bishop–Phelps–Bollobás property for operators. To this purpose the following notion will be useful.