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# Equivalent norms in polynomial spaces and applications

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ABSTRACT

In this paper, equivalence constants between various polynomial norms are calculated. As an application, we also obtain sharp values of the Hardy–Littlewood constants for 2-homogeneous polynomials on $\ell_p^2$ spaces, $2 < p \leq \infty$. We also provide lower estimates for the Hardy–Littlewood constants for polynomials of higher degrees.



## 1. Introduction

Let $\alpha = (\alpha_1, \ldots, \alpha_n) \in (\mathbb{N} \cup \{0\})^n$, and define $|\alpha| := \alpha_1 + \cdots + \alpha_n$. Let $\mathcal{P}(^m\mathbb{K}^n)$ be the finite dimensional linear space of all homogeneous polynomials of degree $m$ on $\mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$). If $\mathbf{x}^\alpha$ stands for the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$ for $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{K}^n$ and $P \in \mathcal{P}(^m\mathbb{K}^n)$, then $P$ can be written as

$$P(\mathbf{x}) = \sum_{|\alpha|=m} a_\alpha \mathbf{x}^\alpha. \tag{1.1}$$

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If $|\cdot|$ is a norm on $\mathbb{K}^n$, then

$$\|P\| := \sup_{x \in B_X} |P(x)|,$$

where $B_X$ is the closed unit ball of the Banach space $X = (\mathbb{K}^n, |\cdot|)$, defines a norm in $\mathcal{P}(^m\mathbb{K}^n)$ usually called polynomial norm. The space $\mathcal{P}(^m\mathbb{K}^n)$ endowed with the polynomial norm induced by $X$ is denoted by $\mathcal{P}(^mX)$. Equivalent norms within the real and complex settings have been the aim of many researchers since the 20th century (see, e.g. [5,6]). Other norms customarily used in $\mathcal{P}(^m\mathbb{K}^n)$ besides the polynomial norm are the $\ell_q$ norms of the coefficients, i.e., if $P$ is as in (1.1) and $q \geq 1$, then

$$|P|_q := \begin{cases} \left(\sum_{|\alpha|=m} |a_\alpha|^q\right)^{\frac{1}{q}} & \text{if } 1 \leq q < +\infty, \\ \max\{|a_\alpha| : |\alpha| = m\} & \text{if } q = +\infty, \end{cases}$$

defines another norm in $\mathcal{P}(^m\mathbb{K}^n)$. It is interesting to observe that the $\ell_q$ norms are equivalent on $\mathbb{K}^n$ and that we have the following well known sharp estimates:

$$|\cdot|_q \leq |\cdot|_s \leq n^{\frac{1}{s}-\frac{1}{q}} |\cdot|_q,$$

for $1 \leq s \leq q$.

The polynomial norm $\|P\|$ is most of the times very difficult to compute, whereas the $\ell_q$ norm of the coefficients $|P|_q$ can be obtained straightforwardly. For this reason it would be convenient to have a good estimate of $\|P\|$ in terms of $|P|_q$. If $\|\cdot\|_p$ represents the polynomial norm of $\mathcal{P}(^m\ell_p^n)$, this paper is devoted to obtain sharp estimates on $\|\cdot\|_p$ $(1 \leq p \leq +\infty)$ by comparison with the norm $|\cdot|_q$ $(1 \leq q \leq +\infty)$. Actually since all norms in finite dimensional spaces are equivalent, the polynomial norm $\|\cdot\|_p$ and the $\ell_q$ norm $|\cdot|_q$ of the coefficients are equivalent in $\mathcal{P}(^m\mathbb{R}^n)$ for all $1 \leq p, q \leq +\infty$, and therefore there exist constants $k > 0$ and $K > 0$ such that

$$k\|P\|_p \leq |P|_q \leq K\|P\|_p, \tag{1.2}$$

for all $P \in \mathcal{P}(^m\mathbb{K}^n)$. If $B_{|\cdot|_q}$ and $B_{\|\cdot\|_p}$ denote, respectively, the closed unit ball of the spaces $(\mathcal{P}(^m\mathbb{K}^n), |\cdot|_q)$ and $(\mathcal{P}(^m\mathbb{K}^n), \|\cdot\|_p)$, then (1.2) shows that the mapping $B_{|\cdot|_q} \ni P \mapsto \|P\|_p$ is bounded by $\frac{1}{k}$ whereas the mapping $B_{\|\cdot\|_p} \ni P \mapsto |P|_q$ is bounded by $K$. Also, the continuity of $P \mapsto \|P\|_p$ and $P \mapsto |P|_q$ over $(\mathcal{P}(^m\mathbb{K}^n), |\cdot|_q)$ and $(\mathcal{P}(^m\mathbb{K}^n), \|\cdot\|_p)$ respectively, together with the fact that the closed unit balls of the spaces $(\mathcal{P}(^m\mathbb{K}^n), |\cdot|_q)$ and $(\mathcal{P}(^m\mathbb{K}^n), \|\cdot\|_p)$ are compact justify, the following definitions:

**Definition 1.1.** If $1 \leq p, q \leq +\infty$ then we define

$$\begin{aligned} k'_{m,n,q,p} &:= \max\left\{\|P\|_p : P \in B_{|\cdot|_q}\right\}, \\ K_{m,n,q,p} &:= \max\left\{|P|_q : P \in B_{\|\cdot\|_p}\right\}. \end{aligned}$$

Since $k'_{m,n,q,p} > 0$, we can define $k_{m,n,q,p} := \frac{1}{k'_{m,n,q,p}}$. Also, we say that $P \in \mathcal{P}(^m\mathbb{K}^n)$ is extremal for $k'_{m,n,q,p}$, $k_{m,n,q,p}$ or $K_{m,n,q,p}$, if $\|P\|_p = k'_{m,n,q,p}|P|_q$, $k_{m,n,q,p}\|P\|_p = |P|_q$ or $|P|_q = K_{m,n,q,p}\|P\|_p$, respectively.

Observe that $k_{m,n,q,p}$ is the biggest $k$ fitting in the first inequality in (1.2) whereas $K_{m,n,q,p}$ is the smallest possible $K$ in the second inequality in (1.2). Also, if a polynomial is extremal for $k'_{m,n,q,p}$, $k_{m,n,q,p}$ or $K_{m,n,q,p}$, then its multiples are also extremal.