J. Math. Anal. Appl. 445 (2017) 1232–1238

Contents lists available at ScienceDirect

Journal of Mathematical Analysis and Applications

www.elsevier.com/locate/jmaa

Convolution operators supporting hypercyclic algebras ☆

Juan Bès [a], J. Alberto Conejero [b,*], Dimitris Papathanasiou [a]

[a] Department of Mathematics and Statistics, Bowling Green State University, Bowling Green, OH 43403, USA
[b] Instituto Universitario de Matemática Pura y Aplicada, Universitat Politècnica de València, València, Spain

ARTICLE INFO

Article history:
Received 24 November 2015
Available online 18 January 2016
Submitted by D. Ryabogin

Dedicated to Professor Richard Aron on the occasion of his 70th birthday

Keywords:
Algebrability
Hypercyclic algebras
Convolution operators
Hypercyclic subspaces
MacLane operator

ABSTRACT

We show that any convolution operator induced by a non-constant polynomial that vanishes at zero supports a hypercyclic algebra. This partially solves a question raised by R. Aron.

1. Introduction

In 2001 Aron, García and Maestre [4] called the attention to the wide range of examples supporting the following "principle": *In many different settings one encounters a problem which, at first glance, appears to have no solution at all. And, in fact, it frequently happens that there is a large linear subspace of solutions to the problem.* This originated growing interest in the study of *large* algebraic structures within nonlinear settings, giving rise to the notions of lineability, spaceability and algebrability, to name a few [1].

This has been the case, for instance, in the study of the set of hypercyclic vectors. It is well known that for any operator T on a topological vector space X, the set

$$HC(T) = \{f \in X : \ \ \{f, Tf, T^2 f, \dots\} \text{ is dense in } X\}$$

☆ This work is supported in part by MICINN and FEDER, Project MTM2013-47093-P, and by GVA, Project ACOMP/2015/005.
* Corresponding author.
E-mail addresses: jbes@bgsu.edu (J. Bès), aconejero@upv.es (J.A. Conejero), dpapath@bgsu.edu (D. Papathanasiou).

http://dx.doi.org/10.1016/j.jmaa.2016.01.029
0022-247X/

of hypercyclic vectors for T is either empty or contains a dense infinite-dimensional linear subspace (but the origin), see [18]. In fact, when $HC(T)$ is non-empty it sometimes contains (but zero) a closed and infinite dimensional linear subspace, while other times the only closed subspaces it contains (but zero) are of finite dimension [8,14]; see also [7, Ch. 8] and [15, Ch. 10].

When X is a topological algebra it is natural to ask whether $HC(T)$ can contain, or must always contain, a subalgebra (but the origin) whenever it is non-empty. Both questions have been answered by considering convolution operators on the space $X = \mathcal{H}(\mathbb{C})$ of entire functions on the complex plane $\mathbb{C}$, endowed with the compact-open topology; that convolution operators (other than scalar multiples of the identity) are hypercyclic was established by Godefroy and Shapiro [13], see also [10,16,5].

Aron et al. [2,3] showed that no translation operator τ_a

$$\tau_a(f)(z) = f(z+a) \quad f \in \mathcal{H}(\mathbb{C}),\ z \in \mathbb{C}$$

can support a hypercyclic algebra, in a very strong way: Indeed, for any positive integer p and any $f \in \mathcal{H}(\mathbb{C})$, the non-constant elements of the orbit of f^p under τ_a are those entire functions for which the multiplicities of their zeros are integer multiples of p. In stark contrast with this operator they also showed that the collection of entire functions f for which every power f^n $(n = 1, 2, \dots)$ is hypercyclic for the operator D of complex differentiation is residual in $\mathcal{H}(\mathbb{C})$.

Later Shkarin [17, Th. 4.1] showed that $HC(D)$ contained both a hypercyclic subspace and a hypercyclic algebra, and with a different approach Bayart and Matheron [7, Th. 8.26] also showed that the set of $f \in \mathcal{H}(\mathbb{C})$ that generate an algebra consisting entirely (but the origin) of hypercyclic vectors for D is residual in $\mathcal{H}(\mathbb{C})$. The abovementioned solutions by Aron et al., Bayart and Matheron, and Shkarin bear the question of which convolution operators on $\mathcal{H}(\mathbb{C})$ support a hypercyclic algebra. In this note we consider the following question:

Question 1 *(Aron).* Let Φ be a non-constant polynomial. Does $\Phi(D)$ support a hypercyclic algebra?

The purpose of this note is to show that the techniques by Bayart and Matheron provide an affirmative answer for the case when $\Phi(0) = 0$:

Theorem 1. *Let Ω be a simply connected planar domain and $\mathcal{H}(\Omega)$ the space of holomorphic functions on Ω endowed with the compact open topology. Let Φ be a non-constant polynomial with $\Phi(0) = 0$. Then the set of functions $f \in \mathcal{H}(\Omega)$ that generate a hypercyclic algebra for $\Phi(D)$ is residual in $\mathcal{H}(\Omega)$.*

2. Proof of Theorem 1

The proof of Theorem 1 follows that of [7, Th. 8.26]. We postpone the proof of Proposition 2 for later.

Proposition 2. *Let Φ be a polynomial with $\Phi(0) = 0$. Then for each pair (U, V) of non-empty open subsets of $\mathcal{H}(\Omega)$ and each $m \in \mathbb{N}$ there exist $P \in U$ and $q \in \mathbb{N}$ so that*

$$\begin{cases} \Phi(D)^q(P^j) = 0 & \text{for } 0 \le j < m, \\ \Phi(D)^q(P^m) \in V. \end{cases} \tag{2.1}$$

Proof of Theorem 1. For any $g \in \mathcal{H}(\Omega)$ and $\alpha \in \mathbb{C}^m$, we let $g_\alpha := \alpha_1 g + \dots + \alpha_m g^m$. Let $(V_k)_k$ be a countable local basis of open sets of $\mathcal{H}(\Omega)$. For each $(k, s, m) \in \mathbb{N}^3$ we let $\mathcal{A}(k, s, m)$ denote the set of $f \in \mathcal{H}(\Omega)$ that satisfy the following property

$$\forall \alpha \in \mathbb{C}^m \text{ with } \alpha_m = 1 \text{ and } \|\alpha\|_\infty \le s,\ \exists q \in \mathbb{N} :\ \Phi(D)^q(f_\alpha) \in V_k. \tag{2.2}$$