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Note

A remark on discontinuity at fixed point

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ABSTRACT

In this note we give one more solution to the open question of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the mapping to be continuous at the fixed point (R. Kannan, 1969 [9]; B.E. Rhoades, 1988 [14]).

1. Introduction

In a nice survey of contractive definitions, Rhoades [13] compared 250 contractive definitions and partially ordered many of these definitions in order to point out the most general fixed point theorem. He also indicated that a large class of contractive definitions does not require the mapping to be continuous in the entire domain. However in all the cases the mapping is continuous at the fixed point (for various contractive definitions see [1–3,9–11]).

In 1988, Rhoades [14] examined in detail the continuity of a large number of contractive mappings at their fixed points and demonstrated that though these contractive definitions do not require the map to be continuous yet the contractive definitions are strong enough to force the map to be continuous at the fixed point. The question whether there exists a contractive definition which is strong enough to generate a fixed point but which does not force the map to be continuous at the fixed point was reiterated by Rhoades in [14] as an existing open problem. The same problem of continuity of multi-valued maps at their fixed points was examined by Hicks and Rhoades [6] in 1992 and through their theorems they showed that the contractive definitions dealt in [6] force the map to be continuous at the fixed point though continuity was

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neither assumed nor implied by the contractive definitions. Jachymski [8] also listed various Meir–Keeler type conditions and established relations between them.

In 1999, Pant [12] proved the following fixed point theorem and obtained the first result that intromit discontinuity at the fixed point:

Theorem 1.1. *If a self-mapping T of a complete metric space (X, d) satisfies the conditions;*

(i). $d(Tx, Ty) \leq \phi(max\{d(x, Tx), d(y, Ty)\})$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$*;*
(ii). *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that*

$$\epsilon < max\{d(x, Tx), d(y, Ty)\} < \epsilon + \delta \text{ implies } d(Tx, Ty) \leq \epsilon$$

then T has a unique fixed point, say z. Moreover, T is continuous at z iff

$$lim_{x \to z} max\{d(x, Tx), d(z, Tz)\} = 0.$$

In this note we give one more solution to the open question of the existence of contractive definitions which are strong enough to generate a fixed point but which do not force the mapping to be continuous at the fixed point [14].

2. Main results

In what follows we shall denote

$$M(x, y) = max\{d(x, y), d(x, Tx), d(y, Ty), [d(x, Ty) + d(y, Tx)]/2\}.$$

Theorem 2.1. *Let (X, d) be a complete metric space. Let T be a self-mapping on X such that T^2 is continuous and satisfy the conditions;*

(i). $d(Tx, Ty) \leq \phi(M(x, y))$, *where* $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *is such that* $\phi(t) < t$ *for each* $t > 0$*;*
(ii). *for a given* $\epsilon > 0$ *there exists a* $\delta(\epsilon) > 0$ *such that* $\epsilon < M(x, y) < \epsilon + \delta$ *implies* $d(Tx, Ty) \leq \epsilon$.

Then T has a unique fixed point, say z, and $T^n x \to z$ for each $x \in X$. Moreover, T is discontinuous at z iff $lim_{x \to z} M(x, z) \neq 0$.

Proof. Let x_0 be any point in X and let $x \neq Tx$. Define a sequence $\{x_n\}$ in X given by the rule $x_{n+1} = T^n x_0 = Tx_n$ and $c_n = d(x_n, x_{n+1})$ for all $n \in \mathbb{N} \bigcup \{0\}$. Then by (i)

$$c_n = d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq \phi(M(x_{n-1}, x_n)) < M(x_{n-1}, x_n) = max\{c_n, c_{n-1}\} = c_{n-1}.$$

Thus $\{c_n\}$ is a strictly decreasing sequence of positive real numbers and, hence, tends to a limit $c \geq 0$. If possible, suppose $c > 0$. Then there exists a positive integer $k \in \mathbb{N}$ such that $n \geq k$ implies

$$c < c_n < c + \delta(c). \tag{2.1}$$

It follows from (ii) and $c_n < c_{n-1}$ that $c_n \leq c$, for $n \geq k$, which contradicts the above inequality. Thus we have $c = 0$.

We shall show that $\{x_n\}$ is a Cauchy sequence. Fix an $\epsilon > 0$. Without loss of generality, we may assume that $\delta(\epsilon) < \epsilon$. Since $c_n \to 0$, there exists $k \in \mathbb{N}$ such that $c_n < \frac{1}{2}\delta$, for $n \geq k$.