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A functional representation of almost isometries ☆

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For each quasi-metric space X we consider the convex lattice $\mathrm{SLip}_1(X)$ of all semi-Lipschitz functions on X with semi-Lipschitz constant not greater than 1. If X and Y are two complete quasi-metric spaces, we prove that every convex lattice isomorphism T from $\mathrm{SLip}_1(Y)$ onto $\mathrm{SLip}_1(X)$ can be written in the form $Tf = c \cdot (f \circ \tau) + \phi$, where τ is an isometry, $c > 0$ and $\phi \in \mathrm{SLip}_1(X)$. As a consequence, we obtain that two complete quasi-metric spaces are almost isometric if, and only if, there exists an almost-unital convex lattice isomorphism between $\mathrm{SLip}_1(X)$ and $\mathrm{SLip}_1(Y)$.

0. Introduction

Suppose you live in a building with no lift. It is clear that your third floor apartment is further from the street than the street is from your apartment, if we think the distance between two places as the time spent for coming from one place to the other one. You may also think about a building with a lift and measure another kind of distance: the time plus the effort – properly weighted. In this case, most people will use the lift when coming home, but not when coming down.

The metrics described above are not symmetric and, moreover, the second one has another strange feature: the ground floor–third floor preferred way (*geodesic*) is different from the third floor–ground floor one.

In this paper we will deal with non-symmetric metrics, called *quasi-metrics*. In the last years there has been an increasing interest about quasi-metric spaces, with applications in a wide variety of topics. We refer to [5,6,8,9] and references therein for further information about the subject. Here we will be interested in some natural transformations of quasi-metric spaces, the so-called *almost isometries* considered in [5]. These are bijections between quasi-metric spaces preserving the triangular function, that is, the quantity

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$\mathrm{Tr}(x, y, z) = d(x, y) + d(y, z) - d(x, z)$. This quantity measures how far the points x, y, z are of achieving equality in the triangle inequality. It is easily seen that, in the case that symmetry holds, almost isometries coincide in fact with isometries.

Our aim is to characterize almost isometries by means of a suitable space of real-valued functions, thus obtaining a theorem of Banach–Stone type in this context. In the symmetric case, it has been shown in [1] that a metric on the space X can be recovered, up to Lipschitz homeomorphisms, by using the lattice structure of the space $\mathrm{Lip}(X)$ of real-valued Lipschitz functions on X. Related results have been obtained in [4] in terms of the unital vector lattice structure of $\mathrm{Lip}(X)$. We refer to the survey paper [3] for further developments about Banach–Stone type theorems. In our case, due to the lack of symmetry, it is natural to consider the space of *semi-Lipschitz* functions in the sense of [9]. A function $f : X \to \mathbb{R}$ defined on a quasi-metric space (X, d) is said to be semi-Lipschitz if there exists a constant $L \geq 0$ such that $f(x) - f(y) \leq L \cdot d_X(x, y)$ for every $x, y \in X$. The space of semi-Lipschitz functions on X will be denoted by $\mathrm{SLip}(X)$. Since we are dealing with almost isometries, we will focus on the subspace $\mathrm{SLip}_1(X)$ of semi-Lipschitz functions with constant $L \leq 1$. It is not difficult to see that $\mathrm{SLip}_1(X)$ is a lattice with the usual pointwise operations. Furthermore, it is closed under convex combinations. As a consequence of our main result, we obtain in Corollary 3.8 a characterization of almost isometries of a quasi-metric space X in terms of the *convex lattice* structure of $\mathrm{SLip}_1(X)$.

The contents of the paper are as follows. In section 1 we review some basic facts about quasi-metrics and almost isometries. It is known that a bijection $\tau : (X, d_X) \to (Y, d_Y)$ between quasi-metric spaces is an almost isometry if, and only if, there exists a function $\phi : X \to \mathbb{R}$ (which is unique up to an additive constant) such that

$$d_Y(\tau(x_1), \tau(x_2)) = d_X(x_1, x_2) + \phi(x_2) - \phi(x_1). \tag{1}$$

We will prove that a bijection between quasi-metric spaces is an almost isometry if, and only if, it preserves the length of closed polygonal paths or, equivalently, the length of any rectifiable closed path. In section 2 we obtain the results about semi-Lipschitz functions which are needed along the paper. As we will see, the extension properties of semi-Lipschitz functions will play an important role. Section 3 is devoted to the main result of the paper. It turns out that, given an almost isometry $\tau : (X, d_X) \to (Y, d_Y)$, the function $\phi : X \to \mathbb{R}$ that *governs* τ in the sense of equation (1) allows us to define a bijection $T : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}_1(X, d_X)$ in the following way: $f \mapsto Tf = f \circ \tau + \phi$. It is not difficult to see that this map preserves order and convex combinations, so we say that it is a *convex lattice isomorphism*. In the opposite direction, note that there are three kinds of natural convex lattice isomorphisms to be considered. Namely:

- $T_1 : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}_1(X, d_X), T_1 f = f \circ \tau$, where τ is an isometry.
- $T_2 : \mathrm{SLip}_1(X, d_X) \to \mathrm{SLip}_1(X, c\, d_X), T_2 f = c \cdot f$, where $c \in (0, \infty)$.
- $T_3 : \mathrm{SLip}_1(X, d'_X) \to \mathrm{SLip}_1(X, d_X), T_3 f = f + \phi$, where $d'(x, x') = d(x, x') + \phi(x') - \phi(x)$.

In Theorem 3.1 we show that every convex lattice isomorphism $T : \mathrm{SLip}_1(Y, d_Y) \to \mathrm{SLip}_1(X, d_X)$ between complete quasi-metric spaces is, in fact, a composition of one of each kind: T is of the form $Tf = c\cdot(f \circ \tau) + \phi$. Finally, in section 4, we give some further consequences and related results.

1. Quasi-metrics

Let X be a set. We will say $d_X : X \times X \to [0, \infty)$ is a *quasi-metric on X* if the following holds:

- $d_X(x, y) = 0$ if and only if $x = y$.
- $d_X(x, y) \leq d_X(x, z) + d_X(z, y)$ for every $x, y, z \in X$.

We say then that the couple (X, d_X) is a quasi-metric space.