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One-sided James' compactness theorem ☆

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A R T I C L E I N F O

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Dedicated to our very dear friend Richard M. Aron

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A B S T R A C T

We present some extensions of classical results that involve elements of the dual of Banach spaces, such as Bishop–Phelp's theorem and James' compactness theorem, but restricting ourselves to sets of functionals determined by geometrical properties. The main result, which answers a question posed by F. Delbaen, is the following: *Let E be a Banach space such that (B_{E^*}, ω^*) is convex block compact. Let A and B be bounded, closed and convex sets with distance $d(A, B) > 0$. If every $x^* \in E^*$ with*

$$\sup(x^*, B) < \inf(x^*, A)$$

attains its infimum on A and its supremum on B, then A and B are both weakly compact. We obtain new characterizations of weakly compact sets and reflexive spaces, as well as a result concerning a variational problem in dual Banach spaces.

1. Introduction

The well-known James' theorem [12] claims that *a bounded closed convex set C in a Banach space E is weakly compact if and only if every $x^* \in E^*$ attains its maximum on C*. There are many reasons why this theorem has attracted the attention of so many researchers: the great number of applications, the search for simpler proofs (see [9,14–16]) and the concern of strengthening it (see [3,19]); in this sense, there are results (see [5,13]) showing that in order to prove that B_E is weakly compact (i.e. E reflexive) we do not have to check that every functional of E^* attains its maximum on B_E, but only those functionals belonging to a certain set which is "big enough" in a topological sense (for instance, a relative weak*-open subset of the unit sphere of E^*).

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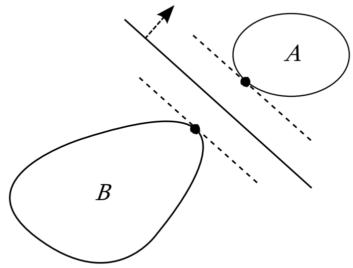

Fig. 1. Hypothesis of Theorem 2.

The aim of this paper is to find sets of functionals which determine the weak compactness of sets in the same spirit of James' compactness theorem, sets determined by geometrical properties. Our motivation is the following question raised by Delbaen:

Question 1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and let A be a bounded convex and closed subset of $\mathbb{L}^1(\Omega, \mathcal{F}, \mathbb{P})$ with $0 \notin A$. Assume that for every $Y \in \mathbb{L}^\infty(\Omega, \mathcal{F}, \mathbb{P})$ with*

$$0 < \inf\{\mathbb{E}[X \cdot Y] \colon X \in A\}$$

we have that this infimum is attained. Is A necessarily uniformly integrable?

The main result of this paper, which gives a positive answer to the previous problem, offers a new characterization of weak compactness in Banach spaces E whose dual unit ball B_{E^*} endowed with the weak*-topology is convex block compact, i.e., every sequence $(x_n^*)_{n\in\mathbb{N}}$ in B_{E^*} has a convex block subsequence which is weak*-convergent.

Theorem 2. *Let E be a Banach space such that (B_{E^*}, ω^*) is convex block compact. Let A and B be bounded, closed and convex sets with distance $d(A, B) > 0$. If every $x^* \in E^*$ with*

$$\sup(x^*, B) < \inf(x^*, A)$$

attains its infimum on A and its supremum on B, then A and B are both weakly compact.

Here, $d(A, B) := \inf\{\|a - b\| : a \in A, b \in B\}$ is the usual distance between sets.

As far as we know, Delbaen's problem was motivated by some questions in the framework of financial mathematics. We hope that some other applications might follow from this last theorem.

The geometrical feature of these functionals (see Fig. 1) motivates the denomination "one-sided" to refer to this new version of James' theorem. This philosophy can be applied to obtain one-sided versions of other classical results such as Bishop–Phelps theorem, for instance if we consider only support points corresponding to supporting functionals satisfying a similar geometrical property. This is discussed in section 2.

In section 3 we revisit the concept of (I)-generation introduced by Fonf and Lindenstrauss [8, p. 159, Definition 2.1], and give in Theorem 6 a one-sided version of their result about the (I)-generation of a set by a James boundary of it. Recall that if $B \subset C$ are subsets of a dual Banach space E^* then B is said to be a *James boundary* of C if for every $x \in E$ we have that the supremum of x on C is attained at some point of B. In our setting, B has this property only for some functionals $x \in E$, which has the advantage that let us consider even unbounded sets. We show in Proposition 7 the equivalence of the one-sided (I)-generation formula given in Theorem 6 and Simons' inequality.

Section 4 is devoted to the proof of Theorem 2. Among the ingredients, we will need a one-sided version of the classical Rainwater–Simons' theorem [7, p. 139, Theorem 3.134], see Theorem 8. The theory developed