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Non-symmetric polarization

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ABSTRACT

Let P be an m-homogeneous polynomial in n-complex variables $x_1, \dots, x_n$. Clearly, P has a unique representation in the form

$$P(x) = \sum_{1 \le j_1 \le \ldots \le j_m \le n} c_{(j_1,\dots,j_m)}\, x_{j_1} \cdots x_{j_m} \,,$$

and the m-form

$$L_P(x^{(1)}, \dots, x^{(m)}) = \sum_{1 \le j_1 \le \ldots \le j_m \le n} c_{(j_1,\dots,j_m)}\, x^{(1)}_{j_1} \cdots x^{(m)}_{j_m}$$

satisfies $L_P(x, \dots, x) = P(x)$ for every $x \in \mathbb{C}^n$. We show that, although L_P in general is non-symmetric, for a large class of reasonable norms $\|\cdot\|$ on $\mathbb{C}^n$ the norm of L_P on $(\mathbb{C}^n, \|\cdot\|)^m$ up to a logarithmic term $(c \log n)^{m^2}$ can be estimated by the norm of P on $(\mathbb{C}^n, \|\cdot\|)$; here $c \ge 1$ denotes a universal constant. Moreover, for the ℓ_p-norms $\|\cdot\|_p$, $1 \le p < 2$ the logarithmic term in the number n of variables is even superfluous.

1. Introduction

It is well-known that for every m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ there is a unique symmetric m-linear form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ such that $L(x, \dots, x) = P(x)$ for all $x \in \mathbb{C}^n$. Uniqueness is an immediate consequence of the well-known *polarization formula* (see e.g. [6, Section 1.1]): For each m-homogeneous polynomial $P : \mathbb{C}^n \to \mathbb{C}$ and each symmetric m-form L on $\mathbb{C}^n$ such that $P(x) = L(x, \dots, x)$ for every $x \in \mathbb{C}^n$, we have for every choice of $x^{(1)}, \dots, x^{(m)} \in \mathbb{C}^n$

$$L\big(x^{(1)}, \dots, x^{(m)}\big) = \frac{1}{2^m m!} \sum_{\varepsilon_k = \pm 1} \varepsilon_1 \cdots \varepsilon_m P\Big(\sum_{k=1}^{m} \varepsilon_k x^{(k)}\Big) .$$

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Moreover, as an easy consequence, for each norm $\|\cdot\|$ on $\mathbb{C}^n$

$$\sup_{\|x^{(k)}\|\le 1} \left|L\big(x^{(1)},\dots,x^{(m)}\big)\right| \le \mathrm{e}^m \cdot \sup_{\|x\|\le 1} |P(x)| \,. \tag{1}$$

Existence can be seen as follows: Every m-homogeneous polynomial $P:\mathbb{C}^n \to \mathbb{C}$ has a unique representation of the form

$$P(x) = \sum_{1\le j_1\le \ldots \le j_m \le n} c_{(j_1,\ldots,j_m)}\, x_{j_1}\cdots x_{j_m}\,.$$

An m-form on $\mathbb{C}^n$ which is naturally associated to P is given by

$$L_P(x^{(1)},\dots,x^{(m)}) := \sum_{1\le j_1\le \ldots \le j_m \le n} c_{(j_1,\ldots,j_m)}\, x^{(1)}_{j_1}\cdots x^{(m)}_{j_m}\,,$$

and the symmetrization $\mathcal{S}L_P$, defined by

$$\mathcal{S}L_P\big(x^{(1)},\dots,x^{(m)}\big) := \frac{1}{m!}\sum_{\sigma} L_P\big(x^{(\sigma(1))},\dots,x^{(\sigma(m))}\big)\,,$$

where the sum runs over all $\sigma \in \Sigma_m$ (the set of all permutations of the first m natural numbers), then is the unique symmetric m-form satisfying $L(x,\dots,x) = P(x)$ for every $x \in \mathbb{C}^n$.

Note that L_P is in general not symmetric. For an arbitrary non-symmetric multilinear form $L : (\mathbb{C}^n)^m \to \mathbb{C}$ and the associated polynomial $P(x) := L(x,\dots,x)$ we have in general no estimate as in (1). Take for example $L : (\mathbb{C}^n)^2 \to \mathbb{C}$ defined by $(x,y) \mapsto x_1y_2 - x_2y_1$. Then $P(x) = L(x,x) = 0$, but $L \ne 0$.

Our purpose is now to establish estimates as in (1) for the multilinear form L_P instead of $\mathcal{S}L_P$. The norms $\|\cdot\|$ we consider on $\mathbb{C}^n$ are 1-unconditional, i.e. $x,y \in \mathbb{C}^n$ with $|x_k| \le |y_k|$ for every k implies $\|x\| \le \|y\|$. Examples are the ℓ_p-norms $\|\cdot\|_p$ for $1 \le p \le \infty$.

Our main result is the following:

Theorem 1.1. *There exists a universal constant $c_1 \ge 1$ such that for every m-homogeneous polynomial $P:\mathbb{C}^n \to \mathbb{C}$ and every 1-unconditional norm $\|\cdot\|$ on $\mathbb{C}^n$*

$$\sup_{\|x^{(k)}\|\le 1} \left|L_P\big(x^{(1)},\dots,x^{(m)}\big)\right| \le (c_1 \log n)^{m^2} \cdot \sup_{\|x\|\le 1} |P(x)| \,. \tag{2}$$

Moreover, if $\|\cdot\| = \|\cdot\|_p$ for $1 \le p < 2$, then there even is a constant $c_2 = c_2(p) \ge 1$ for which

$$\sup_{\|x^{(k)}\|\le 1} \left|L_P\big(x^{(1)},\dots,x^{(m)}\big)\right| \le c_2^{m^2} \cdot \sup_{\|x\|\le 1} |P(x)| \,. \tag{3}$$

Bearing (1) in mind, it suffices to establish the inequality

$$\sup_{\|x^{(k)}\|\le 1} \left|L_P\big(x^{(1)},\dots,x^{(m)}\big)\right| \le c \cdot \sup_{\|x^{(k)}\|\le 1} \left|\mathcal{S}L_P\big(x^{(1)},\dots,x^{(m)}\big)\right|$$

with a suitable constant c. We will prove this inequality by iteration, based on the following theorem. For $1 \le k \le n$ define the partial symmetrization $\mathcal{S}_k L_P : (\mathbb{C}^n)^m \to \mathbb{C}$ of L_P by

$$\mathcal{S}_k L_P\big(x^{(1)},\dots,x^{(m)}\big) := \frac{1}{k!}\sum_{\sigma\in\Sigma_k} L_P\big(x^{(\sigma(1))},\dots,x^{(\sigma(k))},x^{(k+1)},\dots,x^{(m)}\big)\,.$$