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Königs eigenfunction for composition operators on Bloch and H^∞ type spaces

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To Richard Aron on the occasion of his retirement, with very much appreciation

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ABSTRACT

We discuss when the Königs eigenfunction associated with a non-automorphic selfmap of the complex unit disc that fixes the origin belongs to Banach spaces of holomorphic functions of Bloch and H^∞ type. In the latter case, our characterization answers a question of P. Bourdon. Some spectral properties of composition operators on H^∞ for unbounded Königs eigenfunction are obtained.

1. Introduction

The space of analytic functions on the open unit disk $\mathbb{D}$ in the complex plane $\mathbb{C}$ is denoted by $H(\mathbb{D})$. Every analytic selfmap $\varphi\colon \mathbb{D} \to \mathbb{D}$ of the unit disc induces a *composition operator* $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$, given by $C_\varphi f = f \circ \varphi$. For general information of composition operators on classical spaces of analytic functions the reader is referred to the excellent monographs by Cowen and MacCluer [7] and Shapiro [19]. Throughout this paper we assume that $\varphi(0) = 0$ and $0 < |\varphi'(0)| < 1$, and study the *Königs eigenfunction* $\sigma \in H(\mathbb{D})$ for the operator $C_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$. Recall that σ is the unique solution to *Schröder's equation* $\sigma \circ \varphi = \varphi'(0)\sigma$ under the additional assumption $\sigma'(0) = 1$, and can be obtained as the limit of the sequence

$$\sigma_n := \frac{\varphi_n}{\varphi'(0)^n},$$

which converges to σ uniformly on compact subsets of the unit disc as $n \to \infty$. Here φ_n denotes the n-fold composition $\varphi \circ \varphi \circ \cdots \circ \varphi$. For further details on the Königs eigenfunction, for example the fact that σ is

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univalent if and only if the symbol φ is univalent, see chapter 6 in [19]. More generally, if $u \in H(\mathbb{D})$, we consider a *weighted composition operator* $uC_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$, defined by $uC_\varphi(f) = u \cdot (f \circ \varphi)$. Assuming that $u(0) \neq 0$, the eigenvalues of the operator $uC_\varphi : H(\mathbb{D}) \to H(\mathbb{D})$ are given by $\{u(0)\varphi'(0)^n\}_{n=0}^{\infty}$. Each eigenvalue $u(0)\varphi'(0)^n$ has multiplicity one with the function $g\sigma^n$ spanning the corresponding eigenspace, where $g \in H(\mathbb{D})$ is given by

$$g := \lim_{n\to\infty} g_n, \quad \text{where } g_n := \frac{u_{(n)}}{u(0)^n} \text{ and } u_{(n)} := \prod_{k=0}^{n-1} u \circ \varphi_k.$$

Here the convergence is also uniform on compact subsets of $\mathbb{D}$. However, if the composition operator $C_\varphi : \mathcal{A} \to \mathcal{A}$ is restricted to some smaller space $\mathcal{A} \subset H(\mathbb{D})$, then the function σ might not necessarily be an eigenfunction for C_φ since it might be that $\sigma \notin \mathcal{A}$. Therefore our main interest is to determine when σ belongs to different function spaces. The paper focuses on the connection between σ being an element of the function space at hand and spectral properties of the composition operator. Among the spaces we will be interested in are the Bloch-type spaces $\mathcal{B}_\alpha$ and $\mathcal{B}_\alpha^0$ defined by

$$\begin{aligned}
\mathcal{B}_\alpha &= \Big\{ f \in H(\mathbb{D}) \,:\, f(0) = 0,\ ||f||_{\mathcal{B}_\alpha} := \sup_{z\in\mathbb{D}} v_\alpha(z)|f'(z)| < +\infty \Big\} \\
\mathcal{B}_\alpha^0 &= \Big\{ f \in \mathcal{B}_\alpha \,:\, \lim_{|z|\to 1^-} v_\alpha(z)|f'(z)| = 0 \Big\},
\end{aligned}$$

where $v_\alpha(z) := (1-|z|^2)^\alpha$ with $\alpha > 0$. The classical Bloch space $\mathcal{B}_1$ is denoted by $\mathcal{B}$ and the little Bloch space $\mathcal{B}_1^0$ by $\mathcal{B}_0$. The growth spaces H_v^∞ and H_v^0 are defined by

$$\begin{aligned}
H_v^\infty &= \Big\{ f \in H(\mathbb{D}) \,:\, ||f||_{H_v^\infty} := \sup_{z\in\mathbb{D}} v(z)|f(z)| < +\infty \Big\} \\
H_v^0 &= \Big\{ f \in H_v^\infty \,:\, \lim_{|z|\to 1^-} v(z)|f(z)| = 0 \Big\},
\end{aligned}$$

where the weight $v : \mathbb{D} \to \mathbb{R}$ is a continuous, strictly positive and bounded function which is also assumed to be radial, $v(z) = v(|z|)$, and non-increasing with respect to $|z|$. For the spectrum $\sigma_\mathcal{A}(C_\varphi)$ of the composition operator $C_\varphi : \mathcal{A} \to \mathcal{A}$ where $\mathcal{A}$ is any of the spaces $H_{v_\alpha}^\infty$, $H_{v_\alpha}^0$, $\mathcal{B}_\alpha$ or $\mathcal{B}_\alpha^0$ one has [2] that:

$$\Big\{\lambda \in \mathbb{C} : |\lambda| \le r_{e,\mathcal{A}}(C_\varphi)\Big\} \cup \{\varphi'(0)^n\}_{n=1}^\infty \subset \sigma_\mathcal{A}(C_\varphi) \subset \Big\{\lambda \in \mathbb{C} : |\lambda| \le r_{e,\mathcal{A}}(C_\varphi)\Big\} \cup \{\varphi'(0)^n\}_{n=0}^\infty.$$

Notice that the constant function 1 is not an eigenfunction for the Bloch type spaces as it does not lie in the space.

Bourdon [5] succeeded in showing that

$$\sigma \in H_{v_\alpha}^0 \iff |\varphi'(0)| > r_{e,H_{v_\alpha}^0}(C_\varphi), \tag{1.1}$$

and the corresponding result is also true for the H^p-spaces with $p < \infty$ [17]. The natural approach was thus to start by investigating if the result

$$\sigma \in \mathcal{A} \iff |\varphi'(0)| > r_{e,\mathcal{A}}(C_\varphi) \tag{1.2}$$

also holds for the Bloch-type spaces. Since $\mathcal{B}_{\alpha+1}^0 = H_{v_\alpha}^0$ for $\alpha > 0$ and the corresponding norms are equivalent, the result (1.1) of Bourdon gives that

$$\sigma \in \mathcal{B}_{\alpha+1}^0 \iff \sigma \in H_{v_\alpha}^0 \iff |\varphi'(0)| > r_{e,H_{v_\alpha}^0}(C_\varphi) = r_{e,\mathcal{B}_{\alpha+1}^0}(C_\varphi),$$

so we have the following